# Quantum Perfect State Transfer in Cycle Graphs

Elizabeth Athaide[1], Leia Donaway[2], Sam Trombone[3]
[1]Massachusetts Institute of Technology, [2]Swarthmore College, [3]Hamilton College

## Introduction

Our project pertains to a quantum system where at time $t = 0$ we know that with probability one the information is at node $l$. We track the time evolution of the system in order to find a time $T$ where the information is at node $m$ with probability 1 if there exists such a time $T$. We wish to characterize matrices that realize Perfect State Transfer. Specifically we are looking at matrices that represent cycles on $n$ nodes. To categorize these matrices we have examined the spectrum of such matrices. Additionally we have proved that an algorithm that splits paths on $n$ vertices into cycles on $2n-2$ preserves Perfect State Transfer.

## Definitions

1. Consider

$$\mathcal{K} = \begin{bmatrix} 0 & k_1 & 0 & \dots & 0 & x \\ k_1 & 0 & k_2 & 0 & \dots & 0 \\ 0 & k_2 & \ddots & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & & \ddots & 0 & k_{n-1} \\ x & 0 & \dots & 0 & k_{n-1} & 0 \end{bmatrix} \quad (1)$$

This form represents the adjacency matrix of a cycle graph with weights. If $x = 0$ we have a matrix representation of a path on n nodes.

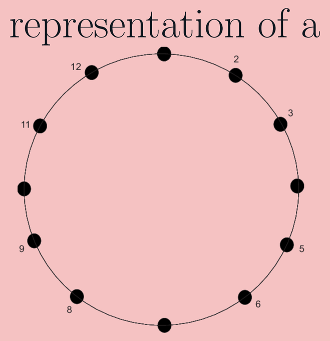


Image of cycle graph, length $n = 12$

Note $\mathcal{K}$ is called a **Periodic Jacobi Matrix**.

2. **Perfect State Transfer**(PST) occurs between nodes $l$ and $m$ if and only if there exists a time $t$ such that

$$e^{itK}e_\ell = \varphi e_m$$

where $\varphi \in$ and $|\varphi| = 1$.

## Theorem 1

Let $\mathcal{K}$ be an $n \times n$ symmetric matrix with real entries and let $\vec{v_k} = (v_{k,1}, v_{k,2}, \dots v_{k,n})$ be the $k$-th eigenvector of $\mathcal{K}$, $1 \le k \le n$, with corresponding eigenvalue $\lambda_k$. The following are necessary and sufficient for Perfect State Transfer to occur between nodes $\ell$ and $m$.

1. $|\vec{v}_{k,\ell}| = |\vec{v}_{k,m}|$ for all $k$ and
2. $\lambda_k - \lambda_{k+1} = \frac{h_k \pi}{T}$, with
   - $h_k \in 2\mathbb{Z}$ if $\frac{\vec{v}_{k,l}}{\vec{v}_{k,m}} = \frac{\vec{v}_{k+1,l}}{\vec{v}_{k+1,m}}$
   - $h_k \in 2\mathbb{Z}+1$ if $\frac{\vec{v}_{k,l}}{\vec{v}_{k,m}} = -\frac{\vec{v}_{k+1,l}}{\vec{v}_{k+1,m}}$

## Theorem 2

Let $\mathcal{K}$ be as in equation (1). If $\mathcal{K}$ does not have a simple spectrum, then $\mathcal{K}$ does not realize perfect state transfer from node 1 to node $n$.

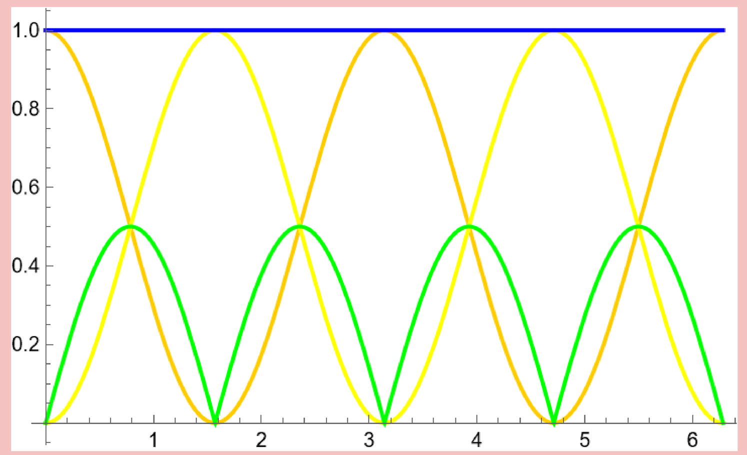
## Example

Consider the matrix $\mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & x \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ x & 0 & 1 & 0 \end{pmatrix}$.

- The differences of eigenvalues $h_i = \lambda_i - \lambda_{i+1}$ must be integer multiples of $\frac{\pi}{T}$
- Then solve for these integers $h_i$:

$$h_1 = -\frac{T(1+x)}{\pi}, h_2 = -\frac{T(-1-x+\sqrt{5-2x+x^2})}{\pi}, h_3 = -\frac{T(1+x)}{\pi}$$

In particular, $\sqrt{5-2x+x^2}$ must be integer-valued. This happens exactly when $x = \pm 1$. Pick $x = 1$.




Node 1: Orange, Node 2: Green, Node 3: Yellow, Node 4: Green

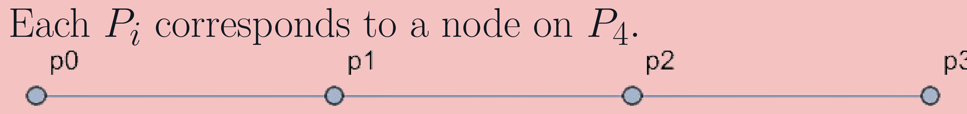
## Path To Cycle Algorithm: $P_4$ to $C_6$

1. Let $\mathcal{P} = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & b_2 & 0 \\ 0 & b_2 & 0 & b3 \\ 0 & 0 & b_3 & 0 \end{pmatrix}$ realize PST. Set Up Eigenvalue equation with $\mathcal{P}$:

$\mathcal{P}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \lambda \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$. $P_0, P_1, P_2, P_3$ are orthogonal polynomials and when they are evaluated at the appropriate $\lambda$ they are the corresponding eigenvector. We use this notation for convenience.

2. We yield the system of equations: $b_1 P_1 = \lambda P_0$, $b_1 P_0 + b_2 P_2 = \lambda P_1$, $b_2 P_1 + b_3 P_3 = \lambda P_2$, $b_3 P_2 = \lambda P_3$.

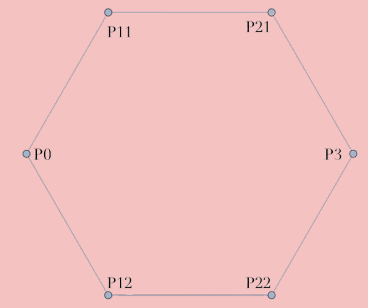
3. Each $P_i$ corresponds to a node on $P_4$.




We will break this path into a cycle by splitting each inner node, $P_i$, into two nodes, $P_i^1, P_i^2$, where,

$$P_i^1 + P_i^2 = P_i \text{ and } P_i^1 = P_i^2.$$

4. Our new cycle can be visualized as




## Theorem 3

The Path to Cycle Algorithm preserves perfect state transfer from node 0 to node $n-1$.

## Orthogonal Polynomials

We can express the eigenvectors of our matrix in terms of **orthogonal polynomials** $P_i$, that satisfy certain conditions:

- $P_i$ of degree $i$ exists for every $i \ge 0$!
- $P_i$, $P_j$ are orthogonal with respect to some positive weight function $w(t)$. In other words:

$$\int_a^b P_i P_j w(t) = 0 \quad \text{if i} \neq \text{j} \qquad \int_a^b P_i P_j w(t) \neq 0 \quad \text{if i = j}$$

For each eigenvalue $\lambda_k$ of $\mathcal{K}$, we can rewrite its eigenvalue equation as

$$\mathcal{K}\begin{pmatrix} P_0(\lambda_k) \\ P_1(\lambda_k) \\ \vdots \\ P_{N-1}(\lambda_k) \end{pmatrix} = \lambda_k \begin{pmatrix} P_0(\lambda_k) \\ P_1(\lambda_k) \\ \vdots \\ P_{N-1}(\lambda_k) \end{pmatrix}$$

## Restatement of Our Conditions

We can rewrite the eigenvector entries corresponding to $\mathcal{K}$ as orthogonal polynomials $\{z_i\}$ evaluated at the eigenvalues of $\mathcal{K}$. To find these $z_i$, we take linear combinations of orthogonal polynomials $\{p_i\}, \{q_i\}$ defined by a three-term recurrence relation, so that

$$z_i(\lambda_k) = \alpha_i p_i(\lambda_k) + \beta_i q_i(\lambda_k).$$

Define the polynomial

$$E(\lambda) := (p_\ell(\lambda) + \beta(\lambda) q_\ell(\lambda))^2 - (p_m(\lambda) + \beta(\lambda) q_m(\lambda))^2.$$

The zeroes of this polynomial include the eigenvalues of $\mathcal{K}$, counted according to multiplicity. Using this fact, we restate our condition 1 from Theorem 1 as follows: If $\mathcal{K}$ has PST from node $\ell$ to node $m$, then it must be the case that

$$E(\lambda) = \det(\mathcal{P} - \lambda_N) G(\lambda)$$

for some rational function $G(\lambda)$

## Future Directions

We hope to characterize periodic Jacobi matrices that realize perfect state transfer.

- Is it always the case that a Hamiltonian that realizes perfect state transfer on a cycle can be reduced to a Jacobi matrix?
- Is our construction the only Hamiltonian that realizes perfect state transfer between nodes 1 and $n$ in a cycle of length $2n-2$?

## Acknowledgements and References

The authors were participants in the 2024 Mathematics REU at UConn. They are grateful for the support of grants from the National Science Foundation (DMS-1950543) and the University of Connecticut. The authors thank Maksym Derevyagin and Rachel Bailey for suggesting the problem and for their mentorship and support.