Perfect State Transfer in Cycle Graphs ♡

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University of Connecticut REU 2024

November 25, 2024

Outline

What we'll show you:

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- Krawtchouk paths proof

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- Splitting paths into cycles
- Our work with orthogonal polynomials thus far
- Plans for future work

Defining K_n

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The underlying Jacobi matrix is mirror symmetric, corresponding to a path weighted by Krawtchouk polynomials, with the first node being node 0 and the last being node N.

Krawtchouk Paths Proposition

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The Krawtchouck-style matrix $\mathcal{K}_N$ *achieves PST from node* 0 *to node* N *at time* $t = \pi$.

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To prove, recall the theorem for conditions sufficient for PST on paths shown by Maksym in lecture.

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Theorem

Let $\mathcal{J}$ be a Jacobi matrix of order N. Then there exists some $t > 0$ and φ such that

$$e^{itJ}e_1 = \varphi e_N \iff$$

1. $\mathcal{J}$ *is mirror symmetric, and*
2. $\lambda_k - \lambda_{k+1} = \frac{(2n_k+1)\pi}{t}, n_k \in \mathbb{Z}$

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Definition

Let $\mathcal{K}'_N$ be defined as corresponding to the following recurrence relation.

$$-xK_n(x,N) = \frac{N-n}{2}K_{n+1}(x,N) - \frac{N}{2}K_n(x,N) + \frac{n}{2}K_{n-1}(x,N)$$

holding for all x if $n = 0, 1, \ldots, N-1$ and for $x = 0, 1, \ldots, N$ for $n = N$.

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holding for all x if $n = 0, 1, \ldots, N-1$ and for $x = 0, 1, \ldots, N$ for $n = N$. $\mathcal{K}'_N$ is mirror symmetric and similar to $\mathcal{K}_N$.

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- This only holds for $x \in 0, 1, \ldots, N-1, \implies 0, 1, \ldots, N$ are the eigenvalues of $\mathcal{K}'_N$.
- We see that the eigenvalue differences for $\mathcal{K}'_N$ are equal to -1, and that $\lambda_k = \lambda'_k + a$ for some constant a, $\lambda_k - \lambda_{k+1} = \lambda'_k - \lambda'_{k+1} - a + a = -1$, so $\lambda_k - \lambda_{k+1} = -1$ is satisfied.

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- Thus, when $t = \pi, \lambda_k - \lambda_{k+1} = \frac{-1\pi}{t}$. This, combined with mirror symmetry, implies by the PST on paths theorem that $\mathcal{K}_N$ realizes PST from node 0 to node N at time $t = \pi$.

□

Orthogonal Polynomials: Review

Recall that we can write our recurrence relation using $\mathcal{K}_N$ in terms of the **orthogonal Kratchouk polynomials**:

$$\mathcal{K}_N \begin{pmatrix} K_0(x) \\ K_1(x) \\ \vdots \\ K_N(x) \end{pmatrix} = -x \begin{pmatrix} K_0(x) \\ K_1(x) \\ \vdots \\ K_N(x) \end{pmatrix}$$

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In fact...

There exists a family of orthogonal polynomials for any Jacobi matrix!

Splitting paths: Initial Algorithm

Path on Four Nodes to Cycle on Six Nodes

If the matrix $\mathcal{P} = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & b_2 & 0 \\ 0 & b_2 & 0 & b3 \\ 0 & 0 & b_3 & 0 \end{pmatrix}$ realizes perfect state transfer then

$$\mathcal{C} = \begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & b_1 \\ \frac{1}{2}b_1 & 0 & b_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & \frac{1}{2}b3 & 0 & 0 \\ 0 & 0 & b_3 & 0 & b_3 & 0 \\ 0 & 0 & 0 & \frac{1}{2}b_3 & 0 & b_2 \\ \frac{1}{2}b_1 & 0 & 0 & 0 & b_2 & 0 \end{pmatrix}$$ also realizes perfect state transfer.

Maksym showed us the process on a P_3 and we made code to generalize it to P_n.

Initial Algorithm

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- We get the system of equations:

$$\begin{aligned} b_1 P_1 &= \lambda P_0 \\ b_1 P_0 + b_2 P_2 &= \lambda P_1 \\ b_2 P_1 + b_3 P_3 &= \lambda P_2 \\ b_3 P_2 &= \lambda P_3 \end{aligned}$$

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- We will break this path into a cycle by splitting each inner node, P_i, into two nodes, P_i^1, P_i^2, where,

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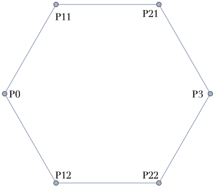
- Our new cycle can be visualized as



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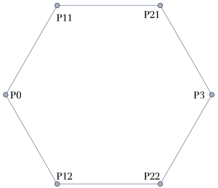

- We will split the weight on the nodes such that the spectrum is preserved.

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\begin{aligned}
b_1 P_1 &= \lambda P_0 \\
\frac{1}{2}(b_1 P_0 + b_2 P_2) &= \lambda P_1^1 \\
\frac{1}{2}(b_2 P_1 + b_3 P_3) &= \lambda P_2^1 \\
b_3 P_2 &= \lambda P_3 \\
\frac{1}{2}(b_2 P_1 + b_3 P_3) &= \lambda P_2^2 \\
\frac{1}{2}(b_1 P_0 + b_2 P_2) &= \lambda P_1^1
\end{aligned}
$$

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$$Q_0 = P_0, \quad Q_1 = P_1^1, \quad Q_2 = P_2^1$$
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- And rewrite our system,

$$b_1 Q_1 + b_1 Q_5 = \lambda Q_0$$
$$\frac{1}{2} Q_0 + b_2 Q_2 = \lambda Q_1$$
$$b_2 Q_1 + \frac{1}{2} Q_3 = \lambda Q_2$$
$$b_3 Q_2 + b_3 Q_4 = \lambda Q_3$$
$$b_2 Q_5 + \frac{1}{2} b_3 Q_3 = \lambda Q_4$$
$$\frac{1}{2} b_1 Q_0 + b_2 Q_4 = \lambda Q_5$$

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$$b_3 Q_2 + b_3 Q_4 = \lambda Q_3$$
$$b_2 Q_5 + \frac{1}{2} b_3 Q_3 = \lambda Q_4$$
$$\frac{1}{2} b_1 Q_0 + b_2 Q_4 = \lambda Q_5$$

Initial Algorithm

- Now we have the matrix $\mathcal{C} = \begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & b_1 \\ \frac{1}{2}b_1 & 0 & b_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & \frac{1}{2}b3 & 0 & 0 \\ 0 & 0 & b_3 & 0 & b_3 & 0 \\ 0 & 0 & 0 & \frac{1}{2}b_3 & 0 & b_2 \\ \frac{1}{2}b_1 & 0 & 0 & 0 & b_2 & 0 \end{pmatrix}$ which has the same eigenvalues as $\mathcal{P}$.

Example

P_5 to C_8

- We will use a P_5 that has weights from the Krawtchouk polynomials.

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P_5 to C_8

- We will use a P_5 that has weights from the Krawtchouk polynomials.
- Start with

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Example

- With the algorithm we get the cycle:

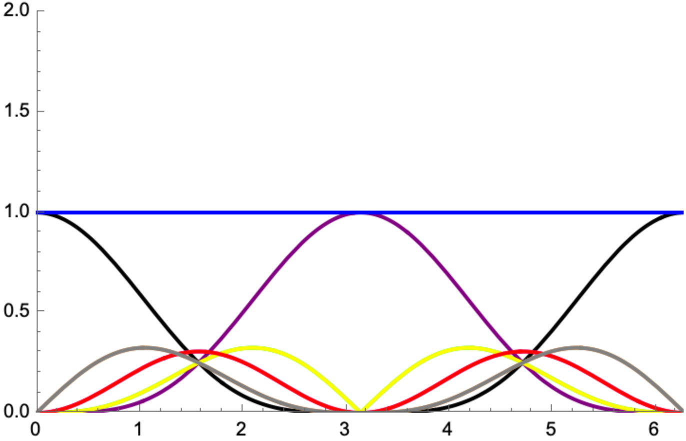
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \end{pmatrix}$$

Example

Do we get PST?

Example

Do we get PST? Yes at $t = \pi$



A Useful Pattern

We did a few examples and saw a pattern:

- $\begin{pmatrix} 0 & b_1 & 0 & b_1 \\ \frac{1}{2}b_1 & 0 & \frac{1}{2}b_2 & 0 \\ 0 & b_2 & 0 & b_2 \\ \frac{1}{2}b_1 & 0 & \frac{1}{2}b_2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & b_1 \\ \frac{1}{2}b_1 & 0 & b_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & \frac{1}{2}b3 & 0 & 0 \\ 0 & 0 & b_3 & 0 & b_3 & 0 \\ 0 & 0 & 0 & \frac{1}{2}b_3 & 0 & b_2 \\ \frac{1}{2}b_1 & 0 & 0 & 0 & b_2 & 0 \end{pmatrix}$

The terms with a coefficient of $\frac{1}{2}$ are in the columns corresponding with nodes that do not have their weight split.

Given $b_1, \ldots, b_n$ this is much easier to code than the algorithm we used to derive these matrices.

Orthogonal Polynomials for a Cycle

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$$\begin{pmatrix} 0 & b_1 & 0 & \dots & 0 & b_N \\ b_1 & 0 & b_2 & \dots & 0 & 0 \\ 0 & b_2 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & & b_{n-1} \\ b_N & 0 & 0 & \dots & b_{n-1} & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ \vdots \\ P_{N-1} \end{pmatrix} = x \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ \vdots \\ P_{N-1} \end{pmatrix}$$

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The first row gives

$$b_1 P_1 + b_N P_{N-1} = x P_0.$$

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If we assume that P_i has degree i, we get a contradiction!

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What if we remove the condition that P_i has degree i? We made it work! We get:

$$\begin{pmatrix} 0 & b_1 & 0 & \dots & 0 & b_N \\ b_1 & 0 & b_2 & \dots & 0 & 0 \\ 0 & b_2 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & & b_{n-1} \\ b_N & 0 & 0 & \dots & b_{n-1} & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_1 \\ P_0 \end{pmatrix} = x \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_1 \\ P_0 \end{pmatrix}$$

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If we set $P_i = P_{N-1-i}$, we no longer have a degree contradiction ☺

A Workaround!

What if we remove the condition that P_i has degree i? We made it work! We get:

$$\begin{pmatrix} 0 & b_1 & 0 & \dots & 0 & b_N \\ b_1 & 0 & b_2 & \dots & 0 & 0 \\ 0 & b_2 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & & b_{n-1} \\ b_N & 0 & 0 & \dots & b_{n-1} & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_1 \\ P_0 \end{pmatrix} = x \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_1 \\ P_0 \end{pmatrix}$$

If we set $P_i = P_{N-1-i}$, we no longer have a degree contradiction ☺
$\implies$ an NxN matrix gives us $\lceil \frac{N}{2} \rceil$ polynomials.

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We can explicitly solve for our polynomials given a matrix!
Let's do the simplest example: $b_i = 1$.

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Our polynomials look like:

$$P_0 = 1, P_1 = x - 1$$
$$P_2 = x^2 - x - 1$$
$$P_3 = x^3 - x^2 - 2x + 1$$
$$P_4 = x^4 - x^3 - 3x^2 + 2x + 1$$

Our Example

It turns out to be pretty hard to check if polynomials are actually orthogonal, but we can observe some properties of these polynomials!

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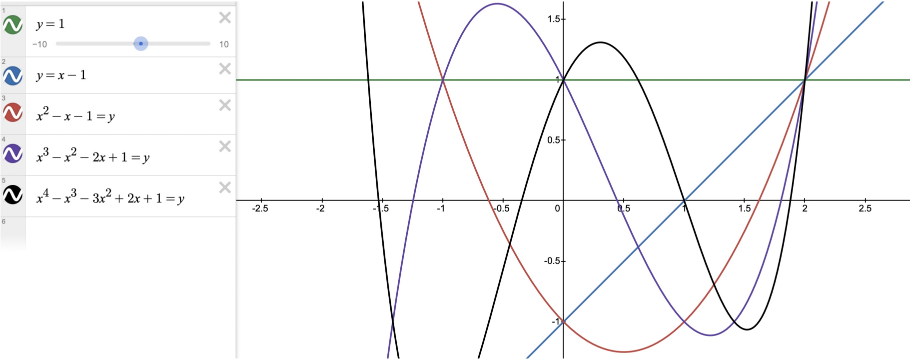

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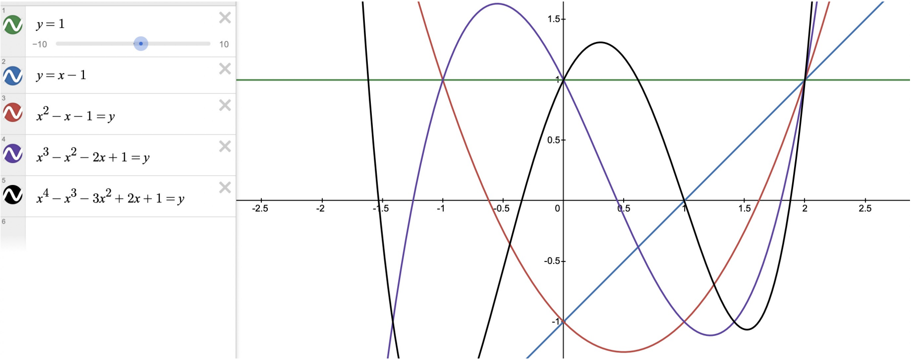

The zeros of these polynomials interlace!
This is evidence that we are on the right track to find some families of orthogonal polynomials.

Future Paths

- Formalizing the splitting algorithm and gaining a better understanding of how it works
- More work with orthogonal polynomials!

Thank you!

(Still to Maksym and Rachel)