Linear Combinations of Chebyshev Polynomials and Early State Exclusion in Weighted Quantum Spin Chains

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Overview

- Introduction and definitions
- Examples and root finding process
- Proof sketch and Tschebyscheff Polynomials
- Something that didn't work

Definition

Definition (Perfect State Transfer)

Let J be a $N+1 \times N+1$ Jacobi matrix. If there exists a time t such that $\mathbf{e}_0^T e^{iJt}\mathbf{e}_0 = 0$ and $|\mathbf{e}_n^T e^{iJt}\mathbf{e}_0| = 1$, then J has **Perfect State Transfer** (PST) at t.

Definition

PST Equiv Conditions

- Recall that a Jacobi matrix has perfect state transfer at time T_0 iff
  - It is mirror symmetric (symmetric across both diagonals)
  - There exists positive integers $N_2, \ldots N_k$ s.t. for its eigenvalues $\lambda_1, \ldots, \lambda_k$, we have

$$\lambda_2 - \lambda_1 = \frac{(2N_2 + 1)\pi}{T_0}, \ldots, \lambda_k - \lambda_{k-1} = \frac{(2N_k + 1)\pi}{T_0} \tag{1}$$

Definition

Definition (Early State Exclusion)

Let J be a $N+1 \times N+1$ Jacobi matrix that has earliest perfect state transfer at time T_0. If there is a time $0 < t < T_0$ such that $\mathbf{e}_0^T e^{iJt} \mathbf{e}_0 = 0$ and $|\mathbf{e}_n^T e^{iJt} \mathbf{e}_0| < 1$, then J has **Early State Exclusion** (ESE) at time t.

Overall Goal

Problem

Our goal is to find weighted paths of length N with Early State Exclusion for infinitely many N.

Examples of PST (no ESE)

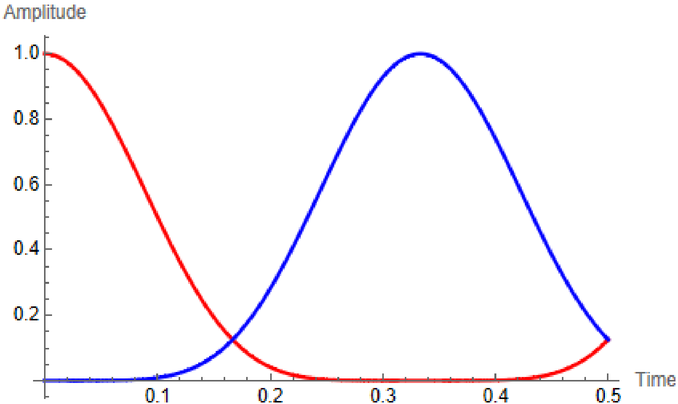

Figure: 4 x 4 Case ($N3 = 1, N3 = 1, N4 = 1$)

Examples of ESE

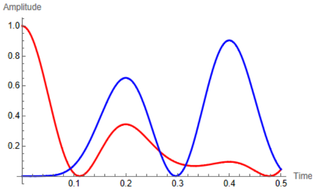

Figure: 4 x 4 Case ($N_2 = 1$, $N_3 = 3$, $N_4 = 1$)

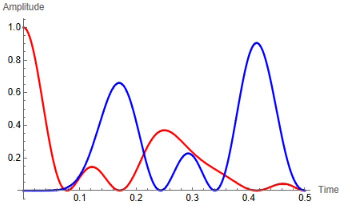

Figure: 5 x 5 Case ($N_2 = 1$, $N_3 = 3$, $N_4 = 3$, $N_5 = 1$)

Root Finding Process

We begin with a set of eigenvalues, $\lambda_0, \lambda_1, ...\lambda_N$ and calculate the normalized eigenvectors of this tridiagonal matrix.

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We begin with a set of eigenvalues, $\lambda_0, \lambda_1, ...\lambda_N$ and calculate the normalized eigenvectors of this tridiagonal matrix.

1. Normalize the eigenvectors so that the matrix which diagonalizes, P, has the inverse $P^{-1} = P^T$.
2. Then, using the spectral theorem, we have that e^{-iDt} equal to:

$$e^{-iDt} = \begin{bmatrix} e^{-i\lambda_0 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-i\lambda_N t} \end{bmatrix}$$

Root Finding Process

3. Then, by making the substitution $t \to i \ln(x + iy)$, we must have $x^2 + y^2 = 1$ (t is real) and we get that

$$e^{-iDt} = \begin{bmatrix} (x+iy)^{\lambda_0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (x+iy)^{\lambda_N} \end{bmatrix}$$

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4. Then we calculate the matrix product according to the usual formula. Since we want to know when $x_1(t)=0$, this becomes

$$(e_0)^T e^{-iJt} e_0 = (e_0)^T P e^{-iDt} P^T e_0 = P(x,y) + iQ(x,y) = 0$$

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5. Then, we have 3 equations, $P(x,y)=0, Q(x,y)=0$, and $x^2+y^2=1$.

Plot

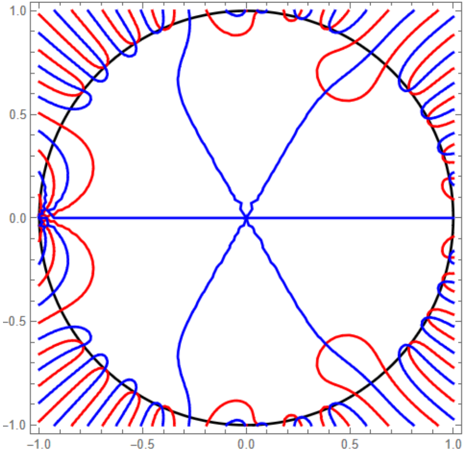

Figure: 7 x 7 Case ($N1 = 1$, $N2 = N3 = N4 = N5 = N6 = 3$, $N7 = 1$)

Earliest P.S.T Lemma

Lemma

Let J be a $n \times n$ mirror symmetric matrix with P.S.T at time π and eigenvalues $\lambda_0, ..., \lambda_n$. Then the earliest time J has perfect state transfer is $\frac{\pi}{gcd(m_0,...,m_{n-1})}$ where $m_i = \lambda_i - \lambda_{i+1}$.

Sufficient Conditions for E.S.E

Theorem

Let n be an even integer and J be a $n \times n$ mirror symmetric matrix with P.S.T at time π. Let $\lambda_0, ..., \lambda_n$ be the eigenvalues of J and let $m_i = \lambda_i - \lambda_{i+1}$. Suppose

1. $\lambda_0, ..., \lambda_n$ *are symmetric around 0*
2. $gcd(m_0, ..., m_{n-1}) = 1$
3. $m_{\frac{n}{2}} > 1$

Then J has E.S.E.

Proof Sketch of Theorem

Proof.

Using Condition 1 we first pair up the negative and positive eigenvalues as and rewrite them as
$\lambda_0 = -x_{\frac{n}{2}}, \dots, \lambda_{\frac{n}{2}} = -x_0$ and
$\lambda_{\frac{n+1}{2}} = x_0, \dots, \lambda_n = x_{\frac{n}{2}}$. Then we may write $\mathbf{e}_0^T e^{iJt}\mathbf{e}_0$ as

$$\sum_{i=0}^{\frac{n}{2}} 2w(x_i)cos(x_s) \tag{2}$$

Observe that each x_s must be of the form $\frac{2k_s+1}{2}$ for some positive integer k_s and that $x_s < x_{s+1}$. By condition 3, $x_0 \neq \frac{1}{2}$. Thus $k_0 > 0$. Then $cos(x_s)$ is a Tschebyschef polynomial of degree k_s using $x = cos(\frac{t}{2})$. □

Proof Sketch of Theorem Continued

Condition 2 combined with the previous lemma implies that we have earliest P.S.T at π. Since cosine is even, if we have P.S.T in the interval $(\pi, 2\pi]$ we would have P.S.T in the interval $[0, \pi)$. Thus the only time we have P.S.T is at π. Following previous work on linear combinations of orthogonal polynomials we know $\sum_{i=0}^{\frac{n}{2}} 2w(x_i)cos(x_s)$ has at least $2k_0 + 1$ roots in $[-1, 1]$. We know one of the roots is at $t = \pi$. The other roots must then correspond to some $t \neq pi$. But we know that we don't have P.S.T since $t \neq \pi$. Thus we have E.S.E.

Something That Didn't Work

It is well known that, for a sequence of orthogonal polynomials, $\{P_n(x)\}_{n=0}^{\infty}$, that $P_n(x)$ must have n zeroes over the support of the measure/weight function, [a,b].

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by design, $P_n(x)$ is orthogonal to any polynomial of degree less than n, so this integral must be zero. However, if R(x) has the same roots as $P_n(x)$, then the integrand is always positive, meaning the integral is non-zero.

Something That Didn't Work: 2

We have proven that it is possible to realize ESE in infinitely many cases of arbitrarily large matrices. However, this relies off of the assumption that the size of the matrix is even. Computationally, this is too strong of a restriction.

Why the problem?

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We end up with an initial state probability amplitude along the lines of:

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Is there a way of repurposing the proof from the previous slide in order to guarantee at least one zero?

Spoiler: No

Suppose that R(t) isn't a monic polynomial which shares the same roots as $x_i(t)$, but is a linear combination of Tchebysheff polynomials:

$$R(t) = \alpha T_1(t) + \beta T_2(t)$$

Then, if R(t) has the same roots as $x_i(t)$, then we can use the same argument with respect to orthogonality to show that there must be at least 1 non-trivial root.

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Then, if R(t) has the same roots as $x_i(t)$, then we can use the same argument with respect to orthogonality to show that there must be at least 1 non-trivial root.

However, by doing some arithmetic, we get that there are some points in the support of the Tschebyschev polynomials so that it is impossible to have only one root

Future Problems (AKA not ours?)

We've seen that the existence of ESE is intimately related to the location of zeroes for Chebyshov polynomials. Given how many fundamental questions in math deal with zeroes, there's a lot of work cut out for whoever continues this work.

Thank you!

Thank you Dr. Maxym and Dr. Bailey! :)