Counting Dirichlet Eigenfunctions

Michael Morris, Jeanette Patel, Shivani Regan, Ty Wick
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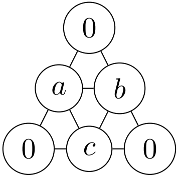
Abstract

In this presentation, we explore the different ways Dirichlet eigenfunctions can be constructed at any given level of the Sierpinski Gasket. This is done by dividing eigenfunctions into smaller groups based on their eigenvalues and generations of birth. This exploration culminates in finding that at the m^{th} level of the Gasket, there are a total of $\frac{3^{m-1}-3}{2}$ Dirichlet eigenfunctions.

Table of Contents

Eigenfunctions on V_1: The Dirichlet Conditions

- Similar to the case of the unit interval ($u(0) = u(1) = 0$), the Dirichlet conditions are applied on the gasket by setting the vertices of V_0 to zero.
- Thus, we can represent a general Dirichlet eigenfunction on V_1 as the following graph:

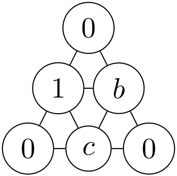


- subject to the eigenfunction equation below [1]:

$$
\begin{aligned}
-\Delta_1 u(x) &= -\sum_{x \sim y} (u(y) - u(x)) \\
&= \deg(x) - \sum_{x \sim y} u(y) = \lambda u(x)
\end{aligned}
$$

Eigenfunctions on V_1: Solving for the Spectrum

- The eigenfunction system of equations (one for each interior vertex x) can be solved in multiple ways. Taking advantage of the linearity of the Laplacian, we normalize the system by setting $a = 1$, and find:



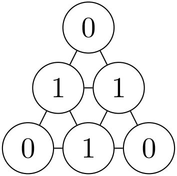
$$\begin{aligned} \lambda + b + c &= 4, \\ (\lambda - 4)b + c &= -1, \quad \Longrightarrow \quad (\lambda - 5)b = (\lambda - 5)c \\ b + (\lambda - 4)c &= -1. \end{aligned}$$

- This creates two cases: $\lambda = 5$ and $\lambda \neq 5$. From some simple algebra we get:

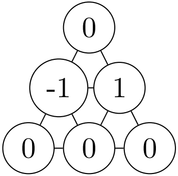
$$\lambda = 5 \implies b + c = -1, \quad \lambda \neq 5 \implies b = c = 1 \implies \lambda = 2.$$

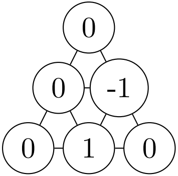
Eigenfunctions on V_1: Solving for the Spectrum

- When $\lambda = 2$, we have the following eigenfunction:



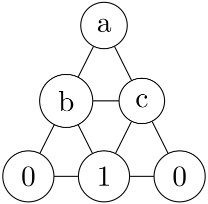
- When $\lambda = 5$, we have the following two eigenfunctions:





Eigenfunctions on V_1: Non-Zero Boundary Points

- An analysis of the following general **non-Dirichlet** eigenfunction is essential to producing the 6-series eigenfunctions on V_m. The general case can be represented as follows:

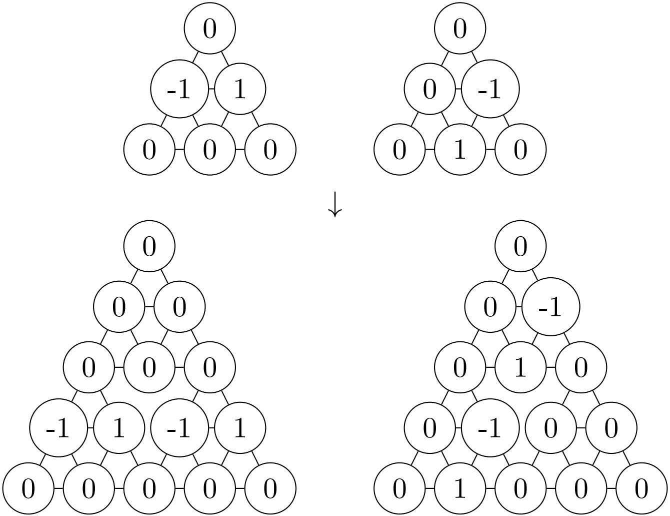


- Through a similar method as before, we obtain the solutions:

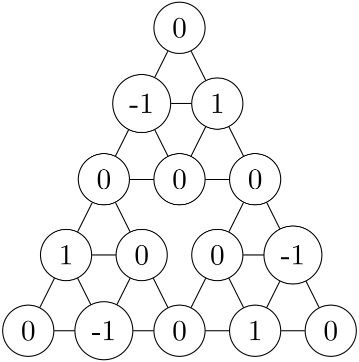
$$b = c = \frac{4-\lambda}{2}, \qquad a = \frac{(\lambda-5)(\lambda-2)}{2}.$$

5-Series: Battery Chain

On any level, there are two battery chains:



Example on V_2:



5-Series: Counting Holes

Each level is three graphs of the previous level glued together, making a new hole in the center

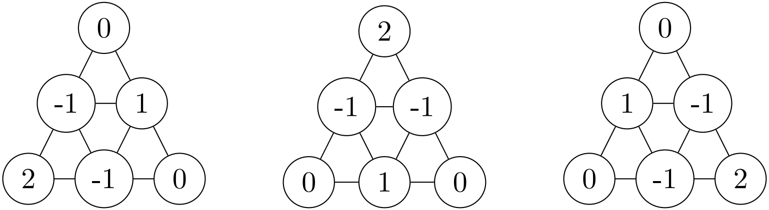
$$H_m = 3H_{m-1} + 1$$

If $H_m = \dfrac{3^m - 1}{2}$ then $H_0 = 0$ and

$$\begin{aligned} H_m =& H_{m-1} + 1 \\ =& 3\left(\frac{3^{m-1} - 1}{2}\right) + 1 \\ =& \frac{3^m - 3}{2} + \frac{2}{2} \\ =& \frac{3^m - 1}{2} \end{aligned}$$

6-Series

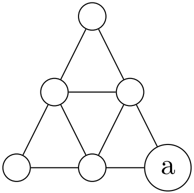
Recall that for V_1, there are no Dirichlet eigenfunctions for $\lambda = 6$. Instead, we have the following three, each with one nonzero boundary:



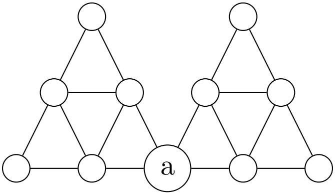
We will now explore two ways to obtain Dirichlet eigenfunctions from the eigenfunctions above.

6-series: Symmetric Extension

Consider the following graph, with Laplacian b at a boundary point with value a.



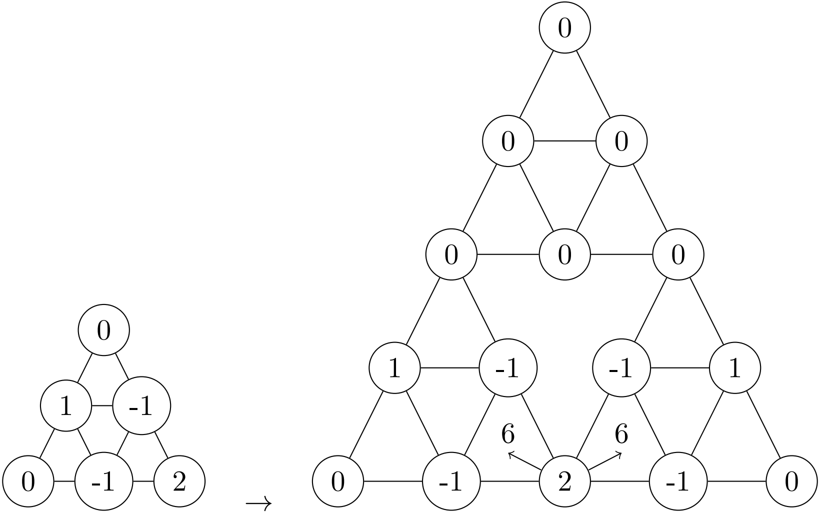
A symmetric extension around a yields



At a, the Laplacian must be $2b$. Thus we require $2b = \lambda a$.

6-series: Symmetric Extension

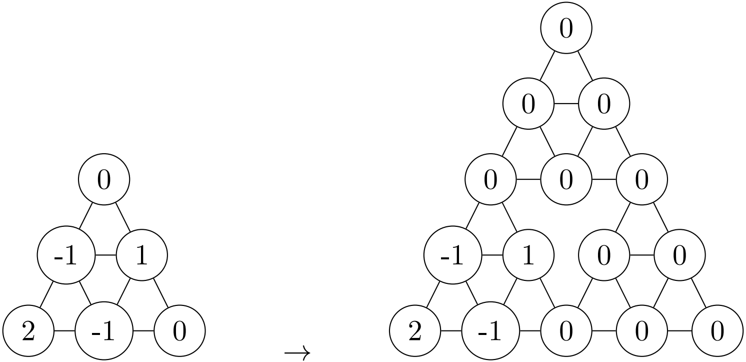
The $2b = \lambda a$ condition is satisfied for $\lambda = 6$ eigenfunctions when reflecting across the non-zero boundary point. For example,



6-series: Basis Element Extensions

We now examine another method of extending eigenfunctions with one non-zero boundary point.

Glue two copies of the zero function to the zero boundary points to obtain three new non-Dirichlet eigenfunctions. For example,



This new eigenfunction still satisfies $2b = \lambda a$ at the non-zero boundary point.

6-series: Basis Element Extensions

As we obtain eigenfunctions by attaching cells, we relate the number of Dirichlet eigenfunctions to the number of interior nodes where those cells intersect.
Thus at level m, we have found

$$3\left(|V_{m-1} \setminus V_0|\right) + 3 = |V_m \setminus V_0|$$

Dirichlet 6-eigenfunctions, all of which have zero Laplacian on the boundary. We will prove later that this is actually all of the Dirichlet 6-eigenfunctions at level m.

Spectral Decimation I

Given an eigenfunction and corresponding eigenvalue at V_{m-1}, we can use spectral decimation to find eigenvalues and eigenfunctions at the V_m level [1].

$$\lambda_m = \frac{5 \pm \sqrt{25 - 4\lambda_{m-1}}}{2}.$$

- The above equation suggests that each $m-1$ eigenvalue bifurcates into two m level eigenvalues with unique eigenfunctions, not counting for spectral decimation.
- Since all eigenvalues are positive, we always have $\lambda_m < 5$.
- That leaves the forbidden eigenvalue 2. This is obtained exactly when $\lambda_{m-1} = 6$.

Thus for V_m, from spectral decimation we get two new eigenfunctions from every $m-1$ non-6-eigenvalue eigenfunction and one new eigenfunction from every 6-eigenvalue eigenfunction.

- The non-6-eigenvalue $m-1$ eigenfunctions come with previous spectral decimation functions (non-forbidden eigenvalues) and 5-eigenvalue functions.
- Giving the following recursive forumula, where D_m is the number of eigenfunctions at level m found through spectral decimation, F_m is the number of 5-series eigenfunctions at level m, and similarly for S_m with 6-series.

$$D_m = 2(D_{m-1} + F_{m-1}) + S_{m-1}$$

Spectral Decimation III

Using the arguments previously given regarding 5-series and 6-series, we claim that the closed form formula is $D_m = \frac{5 \cdot 3^{m-1} - 3}{2}$.

- In the base case, we have $D_1 = \frac{5 \cdot 3^{m-1} - 3}{2} = \frac{5-3}{2} = 1$ as expected.
- Plugging into the recursive formula gives

$$\begin{aligned}
D_m =& 2(D_{m-1} + F_{m-1}) + S_{m-1} \\
=& 2\left(\frac{5 \cdot 3^{m-2} - 3}{2} + \frac{3^{m-2} - 1}{2} + 2\right) + \frac{3^{m-1} - 3}{2} \\
=& 5 \cdot 3^{m-2} - 3 + 3^{m-2} - 1 + 4 + \frac{3^{m-1} - 3}{2} \\
=& \frac{5 \cdot 3^{m-1} - 3}{2}.
\end{aligned}$$

Count Constructed

Having gone through 5-series, 6-series, and spectral decimation, it is a good time to count up how many eigenfunctions we've found for V_m:

- 5-series: $2 + \frac{3^{m-1}-1}{2}$
- 6-series: $\frac{3^m-3}{2}$
- spectral decimation: $\frac{5\cdot 3^{m-1}-3}{2}$

Thus we have found a total of

$$2 + \frac{3^{m-1}-1}{2} + \frac{3^m-3}{2} + \frac{5\cdot 3^{m-1}-3}{2} = \frac{3^{m+1}-3}{2}$$

eigenfuntions at level V_m.

Eigenfunction Bound I

The next question to ask is naturally: Is that all of them? The answer is yes. The easiest way to see why is taking a linear algebra approach.

Recall that a level m Gasket has $\frac{3^{m+1}+3}{2}$ vertices, and thus we can consider it's $\frac{3^{m+1}+3}{2} \times \frac{3^{m+1}+3}{2}$ Laplacian matrix.

Since eigenfunctions/eigenvalues of the Gasket correspond exactly with it's Laplacian matrix, we thus have at most $\frac{3^{m+1}+3}{2}$ eigenfunction/eigenvalue pairs.

Eigenfunction Bound II

But, since we are considering only Dirichlet eigenfunctions, three of our points must be 0. Thus three values of each Dirichlet eigenfunction are 0, and we automatically lose three dimensions to the eigenspace, bringing us down to a maximum of

$$\frac{3^{m+1}+3}{2} - 3 = \frac{3^{m+1}-3}{2}$$

Dirichlet eigenfunctions, which is exactly how many we have found. Thus we have found the number of Dirichlet eigenfunctions on the Gasket at level m.

Sources

Robert S. Strichartz.
Spectrum of the Laplacian, pages 63–90.
Princeton University Press, 2006.