Counting Dirichlet Eigenfunctions

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1 Introduction

In this paper, we explore the different ways Dirichlet eigenfunctions can be constructed at any given level of the Sierpinski Gasket. This is done by dividing eigenfunctions into smaller groups based on their eigenvalues and generations of birth. This exploration culminates in finding that at the m^{th} level of the Gasket, there are a total of $\frac{3^{m-1}-3}{2}$ Dirichlet eigenfunctions.

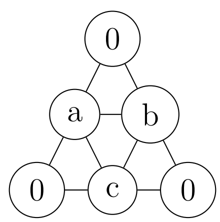
2 Eigenfunctions of the Laplacian on V_1

We can find all Dirichlet eigenfunctions by first finding all such eigenfunctions in the $m = 1$ base case, and then extend to Dirichlet eigenfunctions at some higher level. We know of two types of extensions to construct eigenfunctions in V_m from eigenfunctions in V_{m-1} are spectral decimation and external extension.

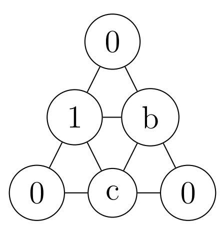
Spectral decimation only constructs Dirichlet eigenfunctions when performed on a Dirichlet eigenfunction. External extensions preserve one of the boundary points, and thus only result in a Dirichlet eigenfunction when performed on an eigenfunction with at least one zero boundary point.

2.1 Dirichlet Eigenfunctions

On V_1, any Dirichlet eigenfunction with eigenvalue λ must be of the form



As any eigenfunction must be non-zero at some point, let $a \neq 0$ without loss of generality. After normalization by a, the graph becomes



As this is an eigenfunction with eigenvalue λ, we use the formula [1]

$$\Delta_1 u(x) = \sum_{x \sim y} (u(x) - u(y))$$

to obtain the following equations

$$\lambda = 4 - b - c$$
$$\lambda b = 4b - c - 1.$$

We solve this linear system through Gaussian elimination

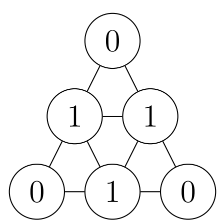
$$\begin{bmatrix} 1 & 1 & 4-\lambda \\ \lambda - 4 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4-\lambda \\ 0 & 5-\lambda & (\lambda-5)(\lambda-3) \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4-\lambda \\ 0 & 1 & 3-\lambda \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix}$$

2.1.1 Suppose $\lambda \neq 5$

Assuming $\lambda \neq 5$ prevents division by zero in the second step above; thus $b = 1$ and $c = 3 - \lambda$. We substitute this into the third equation obtained from the diagram, $\lambda c = 4c - b - 1$, and simplify:

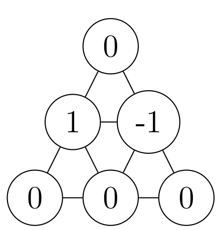
$$\begin{aligned} &\lambda c = 4c - b - 1 \\ \implies &\lambda (3 - \lambda) = 4(3 - \lambda) - 2 \\ \implies &\lambda^2 - 7\lambda + 10 = 0 \\ \implies &(\lambda - 5)(\lambda - 2) = 0 \end{aligned}$$

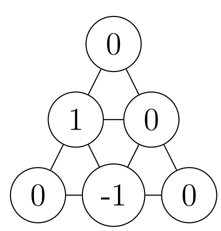
From this computation, either $\lambda = 5$ or $\lambda = 2$. As we already required that $\lambda \neq 5$, it must be that $\lambda = 2$ and $c = 1$. Thus the only Dirichlet eigenfunction with $\lambda \neq 5$ has $\lambda = 2$ and is visualized as



2.1.2 Suppose $\lambda = 5$

When $\lambda = 5$, the third equation $\lambda c = 4c - b - 1$ yields $b + c = -1$. Thus, the eigenspace for Dirichlet eigenfunctions with eigenvalue $\lambda = 5$ is formed from the following basis:

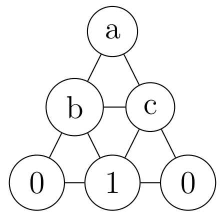




Note, it is easy to see that the third rotation of this eigenfunction is not linearly independent from the two above.

2.2 Non-Zero Boundary Points

We now look at functions that have the potential to extend into Dirichlet eigenfunctions. First, we consider functions with one non-zero boundary point. As rotations of eigenfunctions on a Gasket are still eigenfunctions, we let the normalized case be



The following eigenvalue equations can be read off this diagram:

$$\lambda = 4 - b - c$$
$$\lambda b = 4b - a - c - 1$$
$$\lambda c = 4c - a - b - 1$$

Rearranging the second and third equations yields

$$(4-\lambda)\, b - c = a + 1 = (4-\lambda)\, c - b$$
$$\implies (5-\lambda)\, b = (5-\lambda)\, c$$

Again, we break into cases based on the value of λ.

If $\lambda = 5$, then $a + 1 = -b - c$ implies $a = 0$. This is a contradiction by the non-zero boundary value assumption, so we require $\lambda \neq 5$. It follows that $b = c$.

Substituting into the first equation, $\lambda = 4 - 2b$ implies $b = \dfrac{4-\lambda}{2}$. Thus

$$a + 1 = \frac{(3-\lambda)\,(4-\lambda)}{2} \implies a = \frac{(\lambda-5)\,(\lambda-2)}{2}.$$

Having found a, b, and c in terms of λ, we can now easily construct V_1 eigenfunctions with one non-zero boundary point. This will be useful in a later part of our argument.

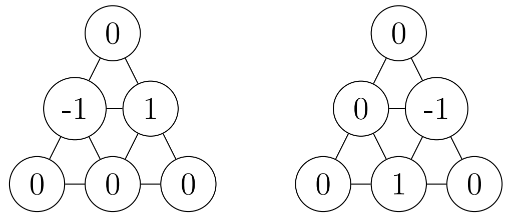
Additionally, through a combinatorial argument later on, it is seen that all Dirichlet eigenfunctions are found without considering external extensions from eigenfunctions with two non-zero boundary points.

Having gone through this V_1 base case, the next few sections of the argument take on the form of an inductive argument, considering different types of Dirichlet eigenfunctions on V_m in relation to the V_{m-1} case.

3 5-series

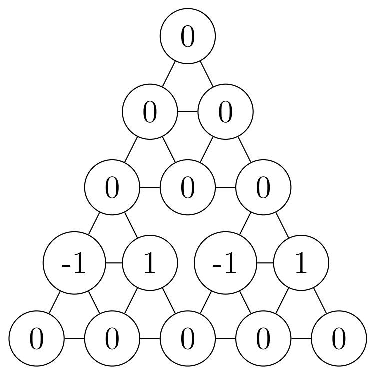
3.1 Basis Element Extensions

We begin with the following eigenfunctions for $\lambda = 5$ on V_1:



Our goal is to construct eigenfunctions of subsequent levels through asymmetric external extensions.

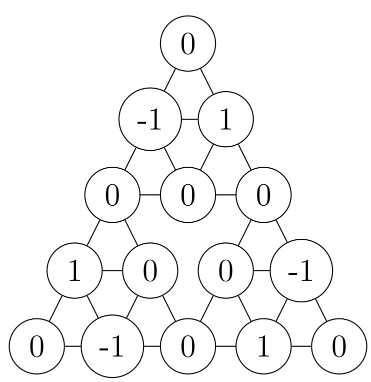
Consider the left eigenfunction and perform an asymmetric reflection around the bottom boundary point to obtain the following battery chain extension:



Likewise, we can do the same for the right eigenfunction and obtain another battery chain expansion. Upon construction of the rest of the Dirichlet 5-eigenvalues, we will find that the third battery chain is not linearly independent.

We can continue to perform battery extensions at each level m; the battery chain eigenfunctions at level m follow the same pattern as above. Thus, for any level m, we will count two battery chain eigenfunctions in our basis of Dirichlet 5-eigenfunctions.

Returning to the left V_1 eigenfunction, we can also perform an asymmetric reflection around the bottom left corner to obtain



This leads to an eigenfunction whose nonzero values circle the central hole of the graph. (We use the term *hole* to denote an upside down triangle in the Gasket, an area that will stay empty at all higher levels.) The three other holes in the diagram above originated as center holes from the lower-level gaskets used for this external construction.

In levels $m \geq 2$, then it must be, by the above construction, that there is a Dirichlet eigenfunction with non-zero values circling the center hole. Then, gluing this Dirichlet eigenfunction with two copies of the zero function gives a new Dirichlet eigenfunction. Thus we have a unique Dirichlet eigenfunction with non-zero values circling each hole that exists in a $m \geq 2$ level sub-Gasket.

We extend V_{m-1} into V_m as above, by connecting V_{m-1} with two cells of identical level, thus creating a new central hole for the resulting diagram on V_m. Because each of the other holes is a center hole at some lower level, it must be that the number of Dirichlet eigenfunctions circling holes is exactly the number of holes which are center holes of some level $m \geq 2$ sub-Gasket. Let the set of these holes be defined as A_m. Then the holes that do not have a circular eigenfunction around them are $\{H_m \backslash A_m\}$ (where H_m is the number of holes on the level m Gasket). Removing these $\{H_m \backslash A_m\}$ holes (and the vertices that create them), leaves eactly the $m-1$ level Gasket. Thus the number of Dirichlet eigenfunctions on V_m circling holes is exactly the number of holes on the $m-1$ level Gasket (H_{m-1}).

Thus, we expect that at level m, there are as many eigenfunctions as there are holes in V_{m-1}, in addition to the two battery chain eigenfunctions.

3.2 Counting Holes

With the number of 5-eigenvalues relying on the number of holes at any given number, we are motivated to find a formula for that value.

We claim that the number of holes H_{m-1} at level $m-1$ is given by $H_{m-1} = \frac{3^{m-1}-1}{2}$.

Proof. Consider the base case m=1. Then $\frac{3^0-1}{2} = 0$, so V_0 has 0 holes. Assume the inductive hypothesis $H_{m-1} = \frac{3^{m-1}-1}{2}$. Again thinking about the level-m Gasket as the composition of three level-$m-1$ Gaskets, we get the following recurrence relation: $H_m = 3H_{m-1} + 1$. Combining this with the inductive hypothesis, we find that

$$H_m = 3\left(\frac{3^{m-1}-1}{2}\right) + 1 = \frac{3^m - 3}{2} + 1 = \frac{3^m - 1}{2}.$$

□

4 Interior Node Count

An important figure to know for the rest of this argument is the number of nodes, and specifically interior nodes, at the m^{th} level. At $m = 1$, there are 6 total nodes of which 3 are interior; at $m = 2$, there are 15 nodes of which 12 are interior.

We claim that the number of interior nodes at level m is

$$|V_m \setminus V_0| = \frac{3^{m+1} - 3}{2}.$$

Proof. We prove this claim by induction. In the base case $m = 1$, $|V_1 \setminus V_0| = \frac{3^{m+1}-3}{2} = \frac{3^2-3}{2} = \frac{6}{2} = 3$ as easily observed on the graph.

Assume the inductive hypothesis $|V_{m-1} \setminus V_0| = \frac{3^m-3}{2}$.

We can think about going from level $m-1$ to level m as taking three copies of the $m-1$-level Gasket and attaching them at the boundary points. Thus, we get the following recurrence relation:

$$|V_m \setminus V_0| = 3|V_{m-1} \setminus V_0| + 3,$$

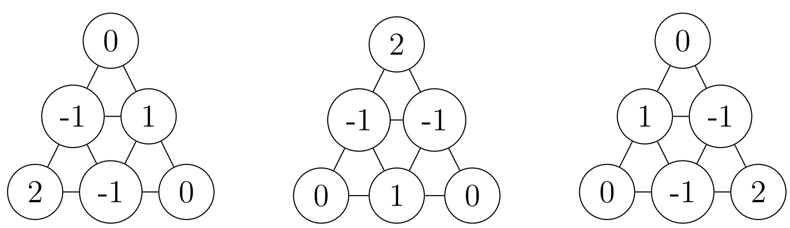
where the additional 3 accounts for where the three $m-1$-level Gaskets attach to each other, converting three boundary points into interior points. Combining this relation with the induction hypothesis yields

$$|V_m \setminus V_0| = 3\left(\frac{3^m-3}{2}\right) + 3 = \frac{3^{m+1}-9+6}{2} = \frac{3^{m+1}-3}{2}$$

as expected. □

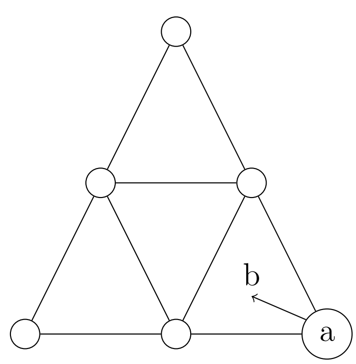
5 6-series

As above in section 2, in V_1, for $\lambda = 6$ there are no Dirichlet eigenfunctions but three eigenfunctions, each with one non-zero boundary condition. They are as follows:

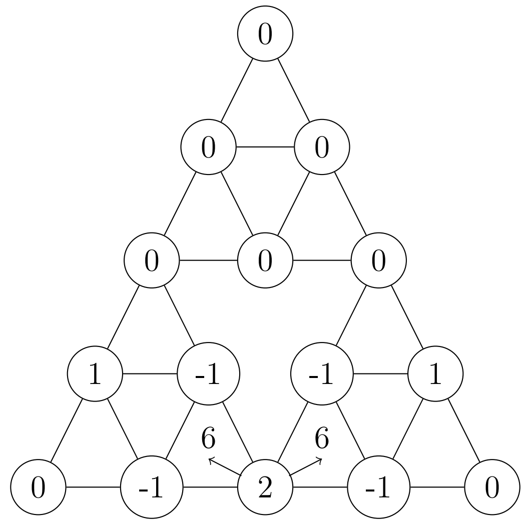


5.1 Symmetric Extension

For a graph with a Laplacian of b at a boundary point with value a as shown below, the symmetric extension about a will result in the Laplacian at a being $2b$. Thus, for the extension to satisfy the eigenvalue equation, it must be that $2b = \lambda a$.



For the $\lambda = 6$ eigenfunctions above, this condition is satisfied when reflected across the non-zero boundary. Thus, the symmetric extension (up to rotations) is



5.2 Basis Element Extensions

From the induction argument, assume V_m has $|V_{m-1} \setminus V_0|$ linearly independent Dirichlet eigenfunctions with the Laplacian equal to zero on the boundary, and an additional three eigenfunctions, each with one non-zero boundary satisfying $2b = \lambda a$ such that the zero boundary points have zero Laplacian.

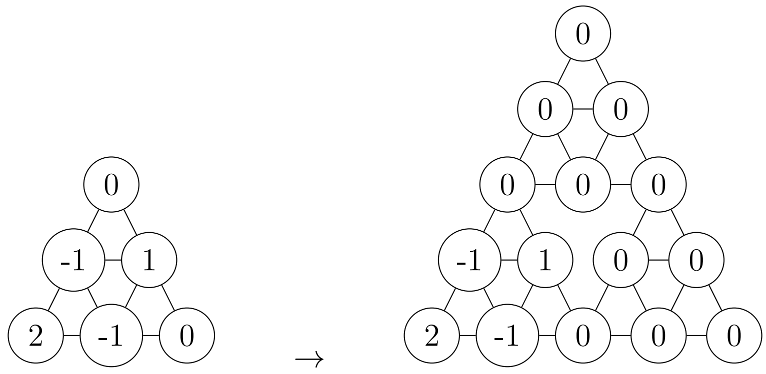
Each of the $m-1$-level Dirichlet eigenfunctions has zero Laplacian on the boundary. Thus, each eigenfunction can be extended in three different ways by gluing two copies of the zero function onto two of the boundary points. Thus each $m-1$-level Dirichlet eigenfunction produces three eigenfunctions at the m-level.

Since the boundary points at the $m-1$-level have a Laplacian of 0, the boundary of the new construction corresponding to a boundary of the lower level must also have zero Laplacian. Moreover, as the lower level eigenfunction was glued with two zero functions, the other two boundary points correspond to a boundary of the zero function and have zero Laplacian. Finally, as the original eigenfunctions were linearly independent, it must be that each of the new constructions with the zero functions glued onto the same places is still linearly independent.

Let β_{m1} be the linearly independent set of the constructed eigenfunctions with the lower-level eigenfunction as the upper cell, and zero functions as the bottom left and right cells. Similarly, let β_{m2} have the constructed eigenfunctions that are non-zero in the bottom left cell, and in the bottom right cell for β_{m3}.

Any element of span$\{\beta_{m1}\}$ must either be the zero function or have nonzero values only in the top third of the gasket, and no element of span$\{\beta_{m2}\}$ has non-zero values in the top third cell. Thus it must be that span$\{\beta_{m1}\} \cap$ span$\{\beta_{m2}\} = \{0\}$. Thus $\beta_{m1} \cup \beta_{m2}$ form a linearly independent set. Similarly, the union of all three bases (β_{m1}, β_{m2}, and β_{m3}) form a linearly independent set.

On the 6-eigenfunctions of V_1, it is true that the two zero boundary points also have a Laplacian of zero. Thus, by gluing two copies of the zero function to those two zero boundary points, we obtain three new non-Dirichlet eigenfunctions.



For each of the three non-Dirichlet eigenfunctions, because they each have two Dirichlet boundary values with zero Laplacian, it follows that the zero eigenfunction can be glued at those Dirichlet points resulting in an eigenfunction with one nonzero boundary value and two boundary values corresponding to boundary points of the zero function (implying zero Laplacian). Because the non-zero boundary point had $2b = \lambda a$ and each a, b, λ stayed unchanged, it follows that the new eigenfunction still has $2b = \lambda a$ at the non-zero boundary point. Because each of the lower-level eigenfunctions had their non-zero boundary point at different positions and extending in this manner preserves the location of the non-zero boundary point, it must be that the new eigenfunctions still have the non-zero values in different positions.

Each of these three non-Dirichlet eigenfunctions can be symmetrically extended (gluing the zero function to complete the construction) to create new Dirichlet eigenfunctions. Because each of the new boundary values corresponds to a zero boundary value of the lower-level eigenfunction or the zero function, it must be that they have zero Laplacian. Each of the previous constructions where found by gluing a Dirichlet eigenfunction with the zero function, thus each of the connections between the different thirds of the construction is zero. In this new construction, the point reflected about is one of these connections however, it is non-zero; thus, the new constructions are independent from the old ones. Each of the three new constructions has the non-zero gluing point at a different location, thus they are linearly independent of each other.

Thus at level $m+1$, we have found

$$3\left(|V_{m-1} \setminus V_0|\right) + 3 = |V_m \setminus V_0|.$$

Dirichlet 6-eigenfunctions, all of which have zero Laplacian on the boundary. We will prove later that this is actually all of the Dirichlet 6-eigenfunctions at level $m+1$.

6 Spectral Decimation Count

We now determine the number of eigenfunctions derived from spectral decimation.

Given an eigenvalue λ_{m-1} at the V_{m-1} level, we can find two eigenvalues at the V_m level using the equation [1]

$$\lambda_m = \frac{5 \pm \sqrt{25 - 4\lambda_{m-1}}}{2}.$$

Using the eigenfunction values at V_{m-1}, the following equation can then be used to fill in the eigenfunction that corresponds to λ_m:

$$u(y_0) = \frac{(4-\lambda_m)(u(x_1)+u(x_2))+2u(x_0)}{(2-\lambda_m)(5-\lambda_m)}$$

where the boundary points of an $m-1$-cell are x_0, x_1, and x_2 are the boundary points of an $m-1$ cell, and $y_0 \in V_m \backslash V_{m-1}$ is opposite to x_0 in that cell.

Thus, each $m-1$ level eigenvalue/eigenfunction pair bifurcates to produce two eigenvalue/eigenfunction pairs at the mth level, due to the $\pm$ in the eigenvalue equation. While counting, however, we must be aware of the possibility of forbidden eigenvalues, which prevents us from carrying out spectral decimation. However, there are a few tricks we can employ to ensure we avoid forbidden eigenvalues.

First, since all eigenvalues are positive, we always have that

$$\lambda_m = \frac{5 \pm \sqrt{25-4\lambda_{m-1}}}{2} < 5.$$

Thus $\lambda_m \neq 5, 6$ when working with spectral decimation.

Now considering $\lambda_m = 2$, we observe how it arises when $\lambda_{m-1} = 6$:

$$\begin{aligned}
\lambda_m = 2 &= \frac{5 \pm \sqrt{25-4\lambda_{m-1}}}{2} \\
4 &= 5 \pm \sqrt{25-4\lambda_{m-1}} \\
-1 &= \pm\sqrt{25-4\lambda_{m-1}} \\
1 &= \sqrt{25-4\lambda_{m-1}} \\
1 &= 25-4\lambda_{m-1} \\
6 &= \lambda_{m-1}.
\end{aligned}$$

We can also verify that $\lambda_m = 3$, a non-forbidden eigenvalue, is the other eigenvalue that arises from $\lambda_{m-1} = 6$. Thus, from each $m-1$ level eigenfunction with eigenvalue 6, we obtain exactly one new eigenvalue λ_m.

We now combine these observations and consider all Laplacian eigenfunctions at level $m-1$.

and specifically which of them have have $\lambda_{m-1} = 6$.

As described above, the number of Dirichlet eigenfunctions found through spectral decimation is twice the number of Dirichlet non-6-series eigenfunctions plus the number of Dirichlet 6-series eigenfunctions in the previous level. By grouping the Dirichlet 2-series eigenfunction on V_1 with the spectral decimation Dirichlet eigenfunctions, it then follows that the number of Dirichlet eigenfunctions from spectral decimation at level m is twice the number of Dirichlet eigenfunctions from spectral decimation at level $m-1$ plus twice the number of Dirichlet 5-series eigenfunctions at level $m-1$ plus the number of Dirichlet 6-series eigenfunctions at level $m-1$. This gives the following recurrence relation (D for spectral decimation, f for 5-series, s for 6-series)

$$D_m = 2\left(D_{m-1} + f_{m-1}\right) + s_{m-1}.$$

Using this grouping of the 2-series with the spectral decimation Dirichlet eigenfunctions, it is then true that in the base case, $D_1 = 1$. By examination, it follows that the closed form solution of this recurrence relation and initial condition is $D_m = \dfrac{5 \cdot 3^{m-1} - 3}{2}$. To confirm this is the correct solution at the initial condition, it follows that $D_1 = \frac{5 \cdot 3^{m-1} - 3}{2} = \frac{5-3}{2} = 1$. Evaluating this solution on the relation also gives

$$\begin{aligned}
D_m =& 2\left(D_{m-1} + f_{m-1}\right) + s_{m-1} \\
=& 2\left(\frac{5 \cdot 3^{m-2} - 3}{2} + \frac{3^{m-2} - 1}{2} + 2\right) + \frac{3^{m-1} - 3}{2} \\
=& 5 \cdot 3^{m-2} - 3 + 3^{m-2} - 1 + 4 + \frac{3^{m-1} - 3}{2} \\
=& 6 \cdot 3^{m-2} + \frac{3^{m-1} - 3}{2} \\
=& \frac{4 \cdot 3^{m-1}}{2} + \frac{3^{m-1} - 3}{2} \\
=& \frac{5 \cdot 3^{m-1} - 3}{2}.
\end{aligned}$$

7 Total Eigenfunctions Found

From above, it is shown that at V_m there are $2 + \dfrac{3^{m-1} - 1}{2}$ 5-series, $\dfrac{3^m - 3}{2}$ 6-series, and $\dfrac{5 \cdot 3^{m-1} - 3}{2}$ spectral decimation Dirichlet eigenfunctions. This means that the number of Dirichlet eigenfunctions we have constructed at V_m thus far is

$$\begin{aligned}
& 2 + \frac{3^{m-1} - 1}{2} + \frac{3^m - 3}{2} + \frac{5 \cdot 3^{m-1} - 3}{2} \\
=& \frac{3^m + 6 \cdot 3^{m-1} - 3}{2} \\
=& \frac{3 \cdot 3^m - 3}{2} \\
=& \frac{3^{m+1} - 3}{2}.
\end{aligned}$$

8 Laplacian Eigenfunction Bound

In trying to determine an upper bound on the number of Laplacian eigenfunctions, a linear algebra approach works well. We can think about an m-level Gasket by considering its Laplacian matrix representation, a $\frac{3^{m+1}+3}{2} \times \frac{3^{m+1}+3}{2}$ matrix. By construction, all Laplacian matrices are positive semi-definite matrices. That is, an $n \times n$ Laplacian matrix will have n non-negative eigenvalues and an n-dimensional eigenspace.

In the case of the m-level Gasket, we have $\frac{3^{m+1}+3}{2}$ eigenvalues with a $\frac{3^{m+1}+3}{2}$-dimensional eigenspace (or equivalently, $\frac{3^{m+1}+3}{2}$ linearly independent eigenvectors/eigenfunctions). We wish to determine how many are Dirichlet eigenfunctions.

For an eigenfunction to be Dirichlet, it must map the three boundary points to 0. That is, given boundary points x_0, x_1, and x_2, all Dirichlet eigenfunctions u must satisfy $u(x_0) = u(x_1) = u(x_2) = 0$. Thus, if we consider the Dirichlet eigenspace as a subspace of the overall eigenspace, we automatically lose three dimensions, or equivalently, three linearly independent eigenfunctions. Thus from this, our upper bound for Dirichlet eigenfunctions at level m is

$$\frac{3^{m+1}+3}{2} - 3 = \frac{3^{m+1}-3}{2}.$$

But as seen in the previous section, we have already found exactly that many Dirichlet eigenfunctions for the m-level Gasket. Therefore, given this upper bound, we must have found all of the Dirichlet eigenfunctions.

9 Conclusion

We determined the Dirichlet eigenfunctions of the m-level Gasket in three different ways, corresponding to different constructions of eigenfunctions:

1. asymmetrical external extensions, used for the 5-series
2. symmetrical external extensions, used for the 6-series
3. (internal) spectral decimation.

By combining the number of Dirichlet eigenfunctions constructed through each method and comparing to a known upper bound for the number of Dirichlet eigenfunctions, we successfully concluded that there exist $\frac{3^{m+1}-3}{2}$ Dirichlet eigenfunctions at the m^{th} level of the Sierpinski-Gasket.

References

[1] Robert S. Strichartz. *Spectrum of the Laplacian*, pages 63–90. Princeton University Press, 2006.