Neumann Eigenfunctions on SG

Kyle Ambrose, Anne Bannon, Noah Dunham, Ranju Iyer, Ben Roark
University of Connecticut

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Abstract

We calculate the Neumann spectrum of the graph Laplacian on the Sierpinski Gasket, and discuss how to derive the Neumann boundary constraints. We write the spectral decimation formula, which is used to obtain eigenfunctions at each level m when $\lambda_{m-1} \neq 2, 5, 6$. We then discuss the eigenfunctions with eigenvalues 5 and 6, which cannot be generated by spectral decimation. A counting argument shows that we have produced every Neumann eigenfunction. Finally, we compute explicitly the Neumann spectra of Δ_1 and Δ_2.

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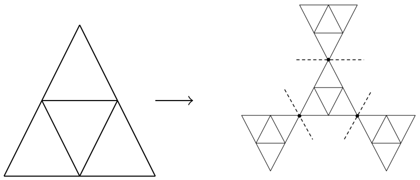
Neumann Boundary Conditions

The Neumann boundary conditions require the normal derivative ∂_n to vanish at the boundary. This condition is satisfied for Γ_m if, when the function is symmetrically reflected across each boundary point, the interior points of this new "mega-gasket" satisfy the eigenvalue equation.



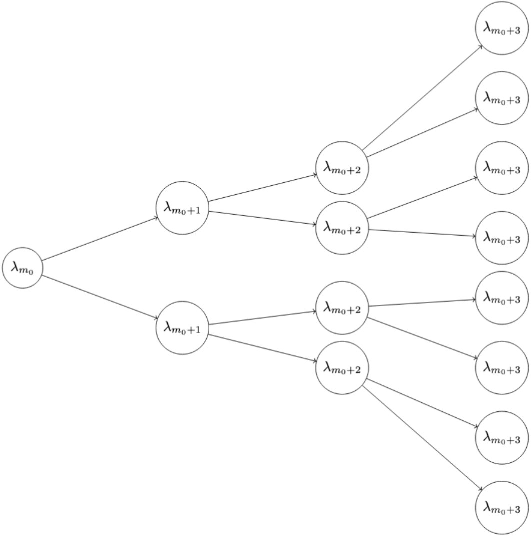
That is, for $q_0, q_1, q_2 \in V_0$, this equation must be true:

$$4u(q_0) - 2u(F_0^m q_1) - 2u(F_0^m q_2) = \lambda_m u(q_0).$$

Continued and Initial Eigenvalues I

Eigenvalues can be divided into two categories: *continued* and *initial*. The continued eigenvalues at level m are obtained through spectral decimation of the eigenvalues at level $m-1$. However, the number of continued eigenvalues is less than the number of total eigenvalues at any given level. The rest of the eigenvalues are *initial eigenvalues* (5 and 6), and their associated eigenfunctions cannot be constructed with spectral decimation. If we take a continued eigenvalue at any level m, it can always be traced back to some level $m_0 < m$, where λ_{m_0} is forbidden. These such eigenvalues are members of a λ_{m_0}-series.

Continued and Initial Eigenvalues II



Spectral Decimation

Given λ_{m-1} and a λ_{m-1}-eigenfunction defined on V_{m-1}, spectral decimation formulas deliver a λ_m-eigenfunction on V_m. For $\lambda_m \neq 2, 5, 6$, we have these relations between subsequent eigenvalues:

$$\lambda_{m-1} = \lambda_m(5 - \lambda_m),$$
$$\lambda_m = \frac{5 \pm \sqrt{25 - 4\lambda_{m-1}}}{2}.$$

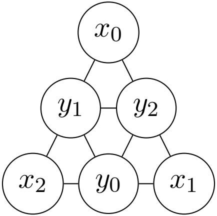
Each eigenvalue at level $m-1$ splits into two eigenvalues at level m (except for $\lambda_{m-1} = 6$, which only leads to $\lambda_m = 3$, and $\lambda_{m-1} = 0$, which only leads to $\lambda_m = 0$).

Spectral Decimation

Using the new λ_m, the new eigenfunction at level m can be constructed using

$$u(y_0) = \frac{(4-\lambda_m)[u(x_1)+u(x_2)]+2u(x_0)}{(2-\lambda_m)(5-\lambda_m)}.$$

Here, x_0, x_1, x_2 are boundary points of the $(m-1)$-cell to which y_0, y_1, y_2 belong, depicted below.



The spectral decimation formulas are obtained from the λ_m eigenvalue equations of $u(y_i)$, and the λ_{m-1} eigenvalue equations of $u(x_i)$.

Constant Eigenfunction

Consider the eigenvalue $\lambda = 0$. We can see that the Neumann condition

$$4u(x_0) = 2u(x_1) + 2u(x_2)$$

is always true for $u(x_0) = u(x_1) = u(x_2)$. Thus, the constant function is an eigenfunction for $\lambda = 0$.

Using spectral decimation, we get that for $\lambda_m = 0$,

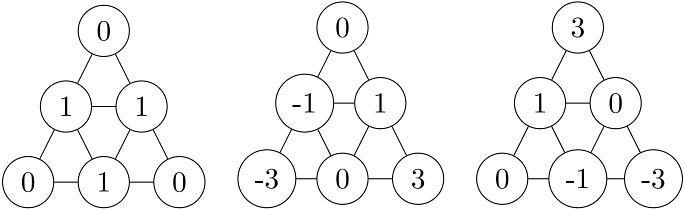
$$\lambda_{m+1} = \frac{5 - \sqrt{25 - 4\lambda_m}}{2} = 0, 5.$$

However, 5 is a forbidden eigenvalue, so this generates only one eigenvalue, $\lambda_{m+1} = 0$.

Thus, $\lambda_m = 0$ is an eigenvalue with multiplicity 1 for all m.

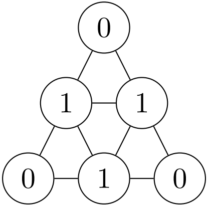
2-Series

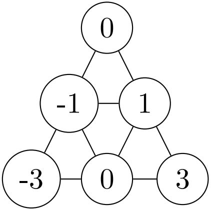
There are no Neumann 2-series for any m. We already know that there are no $\lambda_{m_0} = 2$ Dirichlet eigenfunctions for $m_0 > 1$. We also know all the 2-eigenfunctions can be written as linear combinations of the following three eigenfunctions.

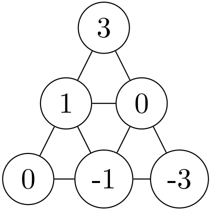


2-Series Proof

To prove there are no Neumann 2-series, we go case by case. Looking at the left most eigenfunction, we see that it is not Neumann anywhere because the Laplacian at the boundary points will be nonzero (when applied with the symmetric extension in mind). The middle and right eigenfunctions have one Dirichlet-Neumann boundary point (0), but these 2-eigenfunctions cannot be extended from that point while still satisfying $-\Delta_2 u = 2u$. So, there are no 2-eigenfunctions at level $m = 2$. An inductive argument tells us there are no 2-eigenfunctions for $m \geq 2$.





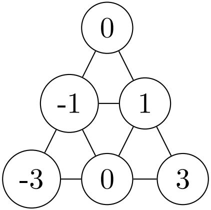


2-Series Proof (cont.)

Also, if we check the Neumann condition

$$4u(q_0) - 2u(F_0^m q_1) - 2u(F_0^m q_2) = \lambda_m u(q_0)$$

for the other two boundary points (-3) and (3), it does not hold. If the boundary points do not satisfy the Neumann condition, they are not Neumann 2-eigenfunctions, and there are thus no 2-eigenfunctions at level $m = 1$. Hence, we have no 2-series in the Neumann spectrum.



5-Series I

The eigenvalue $\lambda_m = 5$ does not arise through spectral decimation. Instead, 5-series eigenfunctions are born at each level $m > 1$. Each such eigenfunction is supported around a hole in the graph Γ_{m-1} and satisfies the Neumann boundary condition due to its symmetric structure.

There are exactly

$$H_m = \frac{3^{m-1} - 1}{2}$$

such holes in Γ_{m-1}, and so this is the number of 5-series eigenfunctions at level m.

Note that for $m = 0$ or $m = 1$, no 5-series eigenfunctions exist because Γ_0 has no holes.

5-Series II

Proof of Hole Count (from last week's presentation):

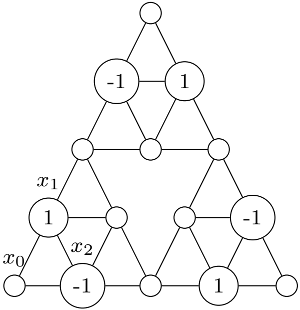
The number of holes in Γ_{m-1} is $\frac{3^{m-1}-1}{2}$.

Base case: $m = 1$. Then Γ_0 has 0 holes, and $\frac{3^0-1}{2} = 0$
Inductive step: Assume $H_m = \frac{3^{m-1}-1}{2}$. Using the recurrence $H_{m+1} = 3H_m + 1$:

$$H_{m+1} = 3\left(\frac{3^{m-1}-1}{2}\right) + 1 = \frac{3^m - 3}{2} + 1 = \frac{3^m - 1}{2}$$

This proves the claim.

5-Series III



If a 5-series eigenfunction forms a cycle around a hole that has nonzero neighbors to a boundary point of the graph, then the symmetry of the loop forces the function values on the neighboring vertices to be $+1$ and -1. This ensures that the Neumann boundary condition holds.

Example: Consider boundary vertex x_0.

$$0 = (4 - \lambda_m)u(x_0) = 2u(x_1) + 2u(x_2) = 2(1) + 2(-1) = 0$$

So the Neumann condition at x_0 is satisfied.

5-Series IV

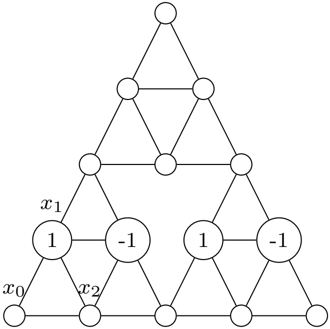
Linear Independence of Loop Eigenfunctions:

Each 5-series eigenfunction is supported around a unique hole in Γ_{m-1}, and there is at least one vertex where it is nonzero and all others are zero.

Because of this, no eigenfunction around one hole can be written as a linear combination of the others. This is because any eigenfunction requires a nonzero value on a vertex where all the others are identically zero. This occurs at every level m.

Therefore, the set of loop 5-series eigenfunctions is linearly independent.

5-Series V



Battery Chains are Not Neumann:
Although battery chain configurations satisfy the Dirichlet condition as we saw in previous presentations, they do not satisfy the Neumann condition. For example, at x_0:

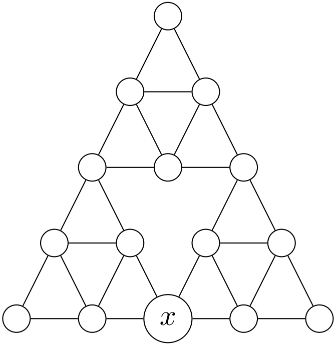
$$(4 - \lambda_m)u(x_0) = 0 \neq 2 = 2(1) + 0 = 2u(x_1) + 2u(x_2)$$

So they are excluded from the count of 5-series Neumann eigenfunctions. This problem occurs for every battery chain at every level m.

6-Series I

There is one 6-eigenfunction at level m for each vertex at level $m-1$. A 6-eigenfunction on V_m can be constructed by first picking a vertex $x \in V_{m-1}$ and letting $u(x) = 2$. There are two cells of level $m-1$ to which x belongs (unless $x \in V_0$, in which case there is only one): $x \in F_w(SG), F_{w'}(SG)$ where $|w| = |w'| = m-1$. Assign values to the vertices in $F_w(SG)$ and $F_{w'}(SG)$ such that they have the same values as the 6-eigenfunctions for $m = 2$, and the 2's match up at x. Finally, let all $x \in V_m$ that are not in the neighboring cells $F_w(SQ), F_{w'}(SG)$ be 0. Any function constructed in this way satisfies $-\Delta_m u = 6u$ for all points in V_m, where the Laplacian of a boundary point is taken considering the symmetric extension.

6-Series II



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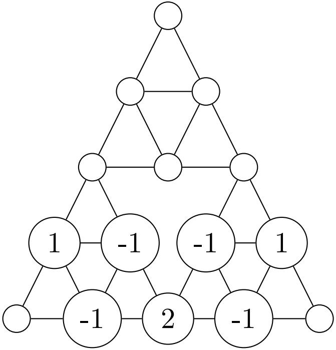
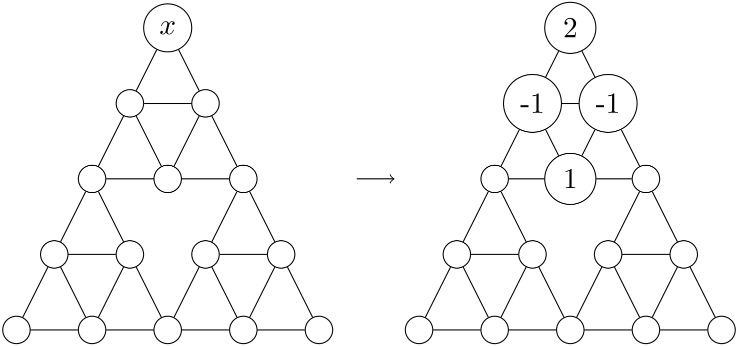


6-Series III



6-Series IV

Linear Independence of 6-Eigenfunctions:

If one eigenfunction is constructed for each vertex $x \in V_{m-1}$ as previously mentioned and all other vertices are set to 0, then this process produces $\#V_{m-1}$ linearly independent eigenfunctions. Linear independence follows from the observation that each of these basis eigenfunctions has $u(x) = 0$ for all but one boundary point, namely the boundary point equaling 2. Thus, no linear combination of these eigenfunctions can produce another one in the basis. All 6-eigenfunctions can be constructed by adding members of this basis. The process repeats at every level m.

6-Series V

The number of vertices at level V_m is defined by

$$\#(V_m) = 3 \cdot \#(V_{m-1}) - 3,$$

with $V_0 = 3$. The general form of $\#(V_m)$ is $\#(V_m) = a3^m + b$, and we solve to find

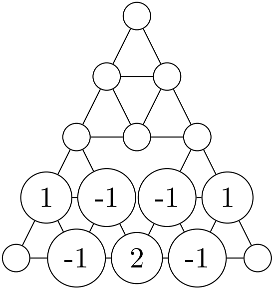
$$\#(V_{m-1}) = \frac{3^m + 3}{2}.$$

This fact will play an important role in our counting argument, where we prove that the Neumann eigenfunctions we have discussed are the only Neumann eigenfunctions.

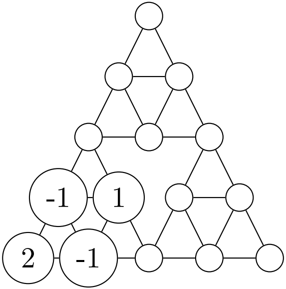
Spectral Decimation Preserves Neumann Conditions

Given that u is a Neumann eigenfunction at level $m-1$, we need to show that spectral decimation preserves the Neumann boundary conditions. To do so, we look at 5-series and 6-series eigenfunctions separately. The constant function is clearly Neumann.

Neumann Conditions: 6-Series and 5-Series



Example I at Level m_0



Example II at Level m_0

If $\lambda_m = 6$, then the boundary points of the eigenfunction will be 0 or 2, so we only need to prove that spectral decimation preserves the Neumann conditions in those two cases. (Because spectral decimation doesn't change V_0.) Also, all 5-series eigenfunctions are 0 at the boundary points.

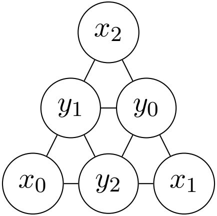
Neumann Condition Preservation

We must show that the Neumann condition is preserved when performing spectral decimation. Assuming the boundary point is x_0 with its neighbors x_1, x_2 at level $m-1$, we must show that if the Neumann condition holds for level $m-1$, then it holds for level m:

$$(4-\lambda_{m-1})u(x_0) = 2u(x_1) + 2u(x_2)$$

$$\implies (4-\lambda_m)u(x_0) = 2u(y_1) + 2u(y_2)$$

This breaks into two cases: $u(x_0) = 0$, or $u(x_0) = 2$.



When $u(x_0) = 0$...

Let $u(x_0) = 0$. Assuming the Neumann condition holds for λ_{m-1}:

$$(4 - \lambda_{m-1})u(x_0) = 2u(x_1) + 2u(x_2)$$
$$0 = u(x_1) + u(x_2)$$

We must show:

$$(4 - \lambda_m)u(x_0) = 2u(y_1) + 2u(y_2)$$
$$0 = u(y_1) + u(y_2)$$

Using the spectral decimation formulas for $u(y_1)$ and $u(y_2)$:

$$u(y_1) = \frac{(4-\lambda_m)[u(x_0) + u(x_2)] + 2u(x_1)}{(2-\lambda_m)(5-\lambda_m)}, \quad u(y_2) = \frac{(4-\lambda_m)[u(x_0) + u(x_1)] + 2u(x_2)}{(2-\lambda_m)(5-\lambda_m)}$$

$$u(y_1) + u(y_2) = \frac{(4-\lambda_m)[u(x_0) + u(x_2)] + 2u(x_1)}{(2-\lambda_m)(5-\lambda_m)} + \frac{(4-\lambda_m)[u(x_0) + u(x_1)] + 2u(x_2)}{(2-\lambda_m)(5-\lambda_m)}$$

$$u(y_1) + u(y_2) = \frac{(4-\lambda_m)[2u(x_0) + u(x_1) + u(x_2)] + 2u(x_1) + 2u(x_2)}{(2-\lambda_m)(5-\lambda_m)}$$

Since $u(x_1) + u(x_2) = 0$, and $u(x_0) = 0$,

$$u(y_1) + u(y_2) = \frac{(4-\lambda_m)[2(0) + 0] + 2(0)}{(2-\lambda_m)(5-\lambda_m)} = 0$$

This completes the proof.

When $u(x_0) = 2$...

Let $u(x_0) = 2$. At the previous level, we know that

$$(4 - \lambda_{m-1})u(x_0) = 2u(x_1) + 2u(x_2),$$

which, plugging in $u(x_0) = 2$ and $\lambda_{m-1} = \lambda_m(5 - \lambda_m)$, yields

$$(4 - \lambda_m)(1 - \lambda_m) = u(x_1) + u(x_2) \Rightarrow 4 - \lambda_m = \frac{u(x_1) + u(x_2)}{1 - \lambda_m}.$$

We want to show that $u(x_0) = 2$ satisfies the Neumann condition at level m:

$$(4 - \lambda_m)u(x_0) = 2u(y_1) + 2u(y_2) \Rightarrow (4 - \lambda_m) = u(y_1) + u(y_2).$$

So, it suffices to show that

$$u(y_1)+u(y_2) = \frac{u(x_1) + u(x_2)}{1 - \lambda_m} \Rightarrow (1-\lambda_m)[u(y_1)+u(y_2)] = u(x_1)+u(x_2).$$

When $u(x_0) = 2$...

Using

$$u(y_1) = \frac{(4-\lambda_m)[2+u(x_2)] + 2u(x_1)}{(2-\lambda_m)(5-\lambda_m)},$$

$$u(y_2) = \frac{(4-\lambda_m)[2+u(x_1)] + 2u(x_2)}{(2-\lambda_m)(5-\lambda_m)},$$

we find that $(1-\lambda_m)[u(y_1)+u(y_2)] =$

$$\frac{(1-\lambda_m)}{(2-\lambda_m)(5-\lambda_m)}[(4-\lambda_m)(2+u(x_2)) + 2u(x_1) + (4-\lambda_m)(2+u(x_1)) + 2u(x_2)]$$

$$= \frac{(1-\lambda_m)}{(2-\lambda_m)(5-\lambda_m)}[4(4-\lambda_m) + (6-\lambda_m)(u(x_1)+u(x_2))].$$

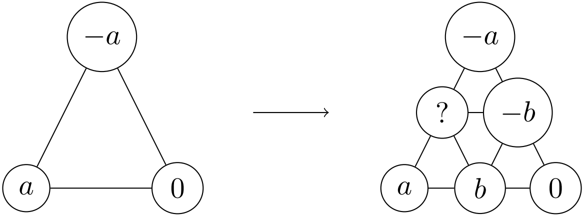
From the fact that u is an eigenfunction at level $m-1$ and $u(x_0) = 2$, we know that the equation $u(x_1)+u(x_2) = (4-\lambda_m)(1-\lambda_m)$ holds. Then the above expression becomes

$$= \frac{(1-\lambda_m)(4-\lambda_m)}{(2-\lambda_m)(5-\lambda_m)}[4 + (6-\lambda_m)(1-\lambda_m)] = \frac{(1-\lambda_m)(4-\lambda_m)(10-7\lambda_m+\lambda_m^2)}{(2-\lambda_m)(5-\lambda_m)}$$

$$= (1-\lambda_m)(4-\lambda_m) = u(x_1)+u(x_2). \blacksquare$$

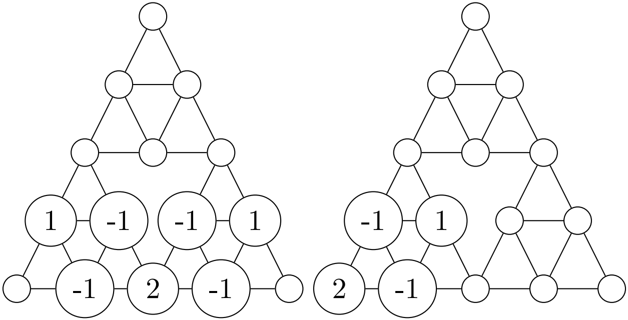
Neumann Conditions: 5-Series Revisited

The preservation of Neumann conditions can be visualized by using (anti-)symmetries.



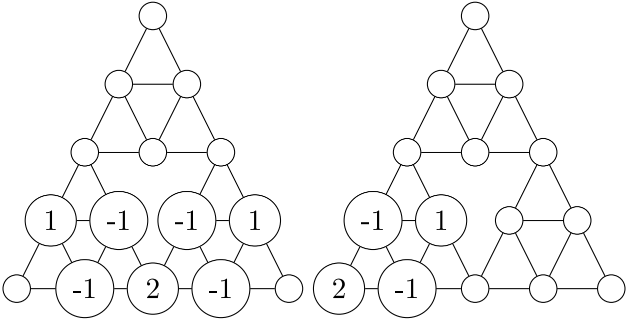
If a boundary value is 0 and the adjacent values are antisymmetric, then the spectral decimation creates antisymmetric conditions on the next level. This is clearly Neumann.
If the vertices adjacent to the boundary are not anti-symmetric, then they are just 0, and spectral decimation preserves this at every level.
Since the boundary points are always 0 for the 5-series, the resulting eigenfunction is still Neumann.

Neumann Conditions: 6-series Revisited I



The two cases (and rotations, reflections, and gluing of 0s) are above. We know that the antisymmetric corners remain Neumann through spectral decimation. The case we need to worry about is the 2 in the corner in the right case, but we already proved that corner remains Neumann.

Neumann Conditions: 6-series Revisited II



Notice that the left cell in figure 2. Is identical to the right cell in figure 1. Additionally, the left cell in figure 1 is the even reflection of this cell. We know spectral decimation on figure 1 results in the Laplacian equation being satisfied, so the same is true for the even reflection around the corner of figure 2.

Counting Eigenfunctions on Γ_m I

To fully determine the Neumann spectrum on Γ_m, we must recover a total of

$$\#(V_m) = \frac{3^{m+1}+3}{2}$$

linearly independent eigenfunctions.

Continued Eigenfunctions from Level $m-1$
At level $m-1$, there are $\frac{3^m+3}{2}$ eigenfunctions. However, not all of these double when continued to level m through spectral decimation. Specifically, the constant 0-series eigenfunction remains unchanged across all levels, and the 6-series eigenfunctions only generate one new branch each.
Therefore, the number of eigenfunctions from level $m-1$ that do double is

$$\frac{3^m+3}{2} - \left(1 + \frac{3^{m-1}+3}{2}\right) = 3^{m-1} - 1$$

These $3^{m-1}-1$ eigenfunctions each contribute two new eigenfunctions to level m.

Counting Eigenfunctions on Γ_m II

Total Neumann Eigenfunction Count at Level m:

The $3^{m-1}-1$ eigenfunctions that double contribute: $2(3^{m-1}-1)$
The constant 0-series eigenfunction contributes 1.
The continued 6-series eigenfunctions from level $m-1$ contribute: $\frac{3^{m-1}+3}{2}$
The new 6-series eigenfunctions born at level m contribute: $\frac{3^m+3}{2}$
The new 5-series eigenfunctions born at level m contribute: $\frac{3^{m-1}-1}{2}$

Adding everything together, we find:

$$2(3^{m-1}-1)+1+\frac{3^{m-1}+3}{2}+\frac{3^m+3}{2}+\frac{3^{m-1}-1}{2}=\frac{3^{m+1}+3}{2}$$

This confirms the expected dimension of the Neumann spectrum on Γ_m.

Neumann Spectrum for Δ_1 and Δ_2 I

For Δ_1, we should have $\frac{3^2+3}{2} = 6$ eigenvalues. Using the information we have discussed, we find that the Neumann spectrum is $\{0, 3, 6\}$ with multiplicities $(1, 2, 3)$. The 3 eigenvalue is gotten from the spectral decimation of the 6 eigenvalue at $m = 0$.
For Δ_2, the Neumann spectrum is $\{0, \frac{5-\sqrt{13}}{2}, 3, \frac{5+\sqrt{13}}{2}, 5, 6\}$ with multiplicities $(1, 2, 3, 2, 1, 6)$. Besides the values gotten from the $0, 5$, and 6 series as discussed earlier, we use spectral decimation on the previous eigenvalue $\lambda_1 = 3$ to get $\lambda = \frac{5\pm\sqrt{13}}{2}$. We can see we have the complete Neuamnn spectrum because we have $\frac{3^3+3}{2} = 15$ eigenvalues.

Neumann Spectrum for Δ_1 and Δ_2 II

The way that the spectrum of Δ_1 creates that of Δ_2 is illustrated below, where the eigenvalues in boxes are initial eigenvalues with generation of birth $m_0 = 2$.