Neumann Boundary Conditions on SG

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Counting Eigenfunctions in the Neumann Spectrum

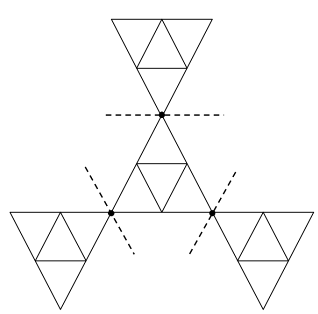
For the Neumann spectrum, we should find a total of $\#V_m = \frac{3^{m+1}+3}{2}$ linearly independent eigenfunctions on each level Γ_m. The Neumann boundary condition requires the normal derivative ∂_n to vanish at the boundary. This condition is satisfied for Γ_m if, when the function is symmetrically reflected across each boundary point (even about the boundary), the boundary points satisfy the eigenvalue equation. If $q_0 \in V_0$ is a boundary point of SG, and $F_0^m q_1, F_0^m q_2$ are its neighbors, the eigenvalue equation evaluated at q_0 becomes

$$\lambda_m u(q_0) = -\Delta_m u(q_0) = 2(u(q_0) - u(F_0^m q_1)) + 2(u(q_0) - u(F_0^m q_2)),$$

which simplifies to:

$$(4 - \lambda_m)u(q_0) = 2u(F_0^m q_1) + 2u(F_0^m q_2).$$

Recall that the contractive mappings F_i satisfy the self-similar identity for SG: $SG = \cup_{i=0}^{2} F_i(SG)$. So, $F_0^m q_1$ and $F_0^m q_2$ are simply the two vertices in V_m that form an edge with q_0.



The Neumann condition can also be thought of as the Mega-Gasket, created through symmetric reflections across the boundary points, satisfying eigenfunction equations at all non-boundary points, where a boundary point is now any vertex with just 2 neighbors.

The Neumann Laplacian can also be expressed as a matrix, which reflects the fact that the boundary points V_0 must satisfy the eigenvalue equation. On V_1, the matrix representation of the Neumann Laplacian is

$$\begin{bmatrix} 4 & 0 & 0 & 0 & -2 & -2 \\ 0 & 4 & 0 & -2 & 0 & -2 \\ 0 & 0 & 4 & -2 & -2 & 0 \\ 0 & -2 & -2 & 4 & -1 & -1 \\ -2 & 0 & -2 & -1 & 4 & -1 \\ -2 & -2 & 0 & -1 & -1 & 4 \end{bmatrix},$$

if the columns/rows are ordered so that the first three columns/rows correspond to the boundary points $q_i \in V_0$ and the last three columns/rows correspond to the interior points $y_i \in V_1 \setminus V_0$. When multiplied against the vector $u|_{V_1}$, it is clear why this representation makes sense:

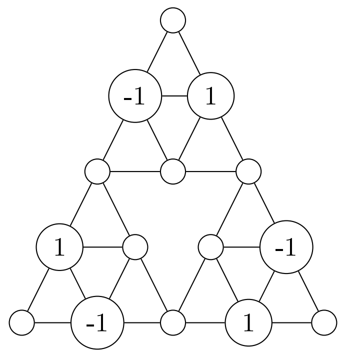
$$\begin{bmatrix} 4 & 0 & 0 & 0 & -2 & -2 \\ 0 & 4 & 0 & -2 & 0 & -2 \\ 0 & 0 & 4 & -2 & -2 & 0 \\ 0 & -2 & -2 & 4 & -1 & -1 \\ -2 & 0 & -2 & -1 & 4 & -1 \\ -2 & -2 & 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u(q_0) \\ u(q_1) \\ u(q_2) \\ u(y_0) \\ u(y_1) \\ u(y_2) \end{bmatrix} = \begin{bmatrix} -\Delta_1 u(q_0) \\ -\Delta_1 u(q_1) \\ -\Delta_1 u(q_2) \\ -\Delta_1 u(y_0) \\ -\Delta_1 u(y_1) \\ -\Delta_1 u(y_2) \end{bmatrix}.$$

Here, $-\Delta_1 u(q_i)$ is the Laplacian at $q_i \in V_0$, once the function has been evenly reflected across x_i.

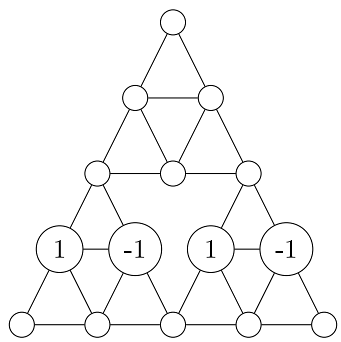
Initial Eigenfunctions Born at Level m

5-Series

The forbidden eigenvalue 5 cannot arise as a result of the spectral decimation formula. Rather, at each level m, 5-series eigenfunctions are born. Each such eigenfunction corresponds to a hole in the graph Γ_{m-1}, and satisfies the Neumann condition due to its symmetric structure. There are $\frac{3^{m-1}-1}{2}$ such holes. Battery-chain configurations satisfying the Dirichlet conditions are excluded from this count, as they do not satisfy the Neumann conditions. Note that there cannot exist 5-eigenfunctions for $m = 0$ or $m = 1$, because only for $m > 1$ does the preceding graph Γ_{m-1} have a hole.



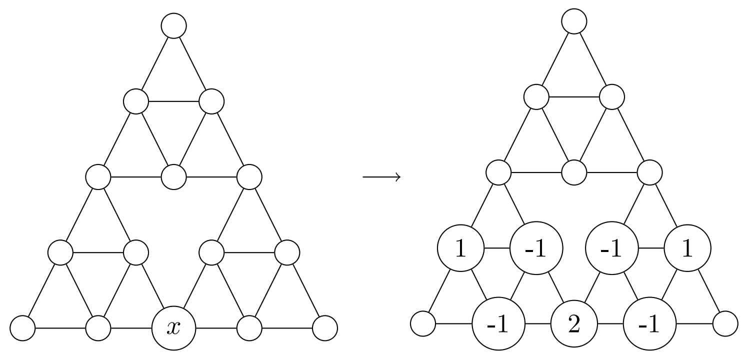
The 5-series eigenfunctions which are loops around the holes satisfy the Neumann conditions at the boundary. $(4-\lambda_m)u(x_0) = 2u(x_1) + 2u(x_2) \implies 0 = 2(1) + 2(-1)$. The loops form a linearly independent set since each is a localized function around a separate hole. Any linear combination of the corresponding eigenfunctions remains localized within their respective regions and cannot generate an eigenfunction with around a different hole.

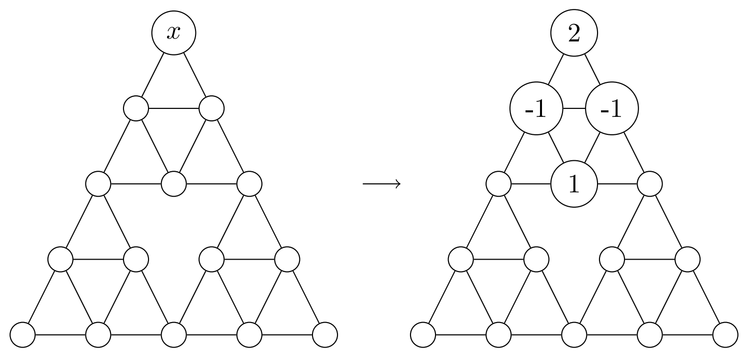


The battery chains do not satisfy the Neumann boundary condition. At the boundaries, $(4-\lambda_m)u(x_0) = 0 \neq 2u(x_1) + 2u(x_2) = 2(1) + 0 = 2$.

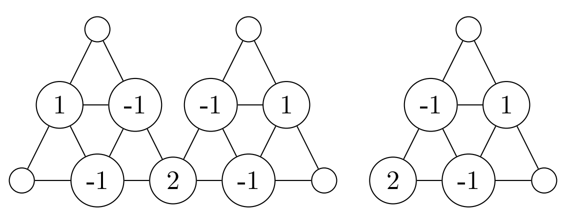
6-series

There is one 6-eigenfunction at level m for each vertex at level $m-1$, so the multiplicity of $\lambda_m = 6$ is $M_m(6) = \#V_{m-1} = \frac{3^m+3}{2}$. A 6-eigenfunction on V_m can be constructed by first picking a vertex $x \in V_{m-1}$ and letting $u(x) = 2$. There are two cells of level $m-1$ to which x belongs (unless $x \in V_0$, in which case there is only one): $x \in F_w(SG), F_{w'}(SG)$ where $|w| = |w'| = m-1$. Assign values to the vertices in $F_w(SG)$ and $F_{w'}(SG)$ such that they have the same values as the 6-eigenfunctions for $m = 2$, and the 2's match up at x. Finally, let all $x \in V_m$ that are not in the neighboring cells $F_w(SQ), F_{w'}(SG)$ be 0. Any function constructed in this way satisfies $-\Delta_m u = 6u$ for all points in V_m, where the Laplacian of a boundary point is taken considering the symmetric extension. Some examples of this kind of construction for $m = 2$ follow.



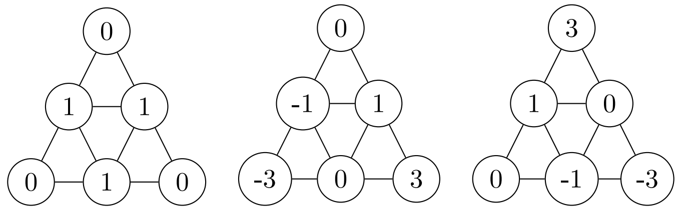


If one eigenfunction is constructed for each vertex $x \in V_{m-1}$ as previously mentioned and all other vertices are set to 0, then this process produces $\#V_{m-1}$ linearly independent eigenfunctions. Linear independence follows from the observation that each of these basis eigenfunctions has $u(x) = 0$ for all but one boundary point, namely the boundary point equaling 2. Thus, no linear combination of these eigenfunctions can produce another one in the basis. All 6-eigenfunctions can be constructed by adding members of this basis. The process repeats at every level m.



2-series

There are no Neumann 2-series for any m. We know that there are no $\lambda_{m_0} = 2$ Dirichlet eigenfunctions for $m_0 > 1$. We also know all the 2-eigen functions must be within these possibilities.



To prove there are no Neumann 2-series, we go case by case. Looking at the left most eigenfunction, we see that it is not Neumann. The middle and right eigen functions have one Dirichlet-Neumann boundary point (0) but we know these 2 eigen functions cannot be extended from that point. Also checking the Neumann condition (3.3.1) in the book for the other two boundary points (-3) and (3) it does not satisfy. If the boundary points do not satisfy the Neumann condition, they are not Neumann 2-eigen functions. Hence we have no 2-series in the Neumann spectrum.

Constant

There is always one constant function with $\lambda_m = 0$, which does not double at the next level. This is evident because the spectral decimation formula for $\lambda_{m-1} = 0$ yields $\lambda_m = 0, 5$ and five is forbidden. ($\lambda_m = \frac{5 \pm \sqrt{25 - 4\lambda_{m-1}}}{2} = \frac{5 \pm 5}{2} = 0, 5$). Thus, only a 0-eigenfunction continues to the next level Γ_m.

Continued Eigenfunctions from Level $m-1$

From level $m-1$, there are

$$\frac{3^m + 3}{2}$$

eigenfunctions. However, not all of these double through spectral decimation when continuing to level m. Two types of eigenfunctions do not double. The constant 0-series eigenfunction remains unchanged across all levels. The 6-series eigenfunctions only generate one branch through spectral decimation.

Therefore, the number of eigenfunctions from level $m-1$ that do double at level m is:

$$\frac{3^m + 3}{2} - \left(1 + \frac{3^{m-1} + 3}{2}\right) = 3^{m-1} - 1.$$

These $3^{m-1} - 1$ eigenfunctions double in level m through spectral decimation.

Total Count of Neumann Eigenfunctions at Level m

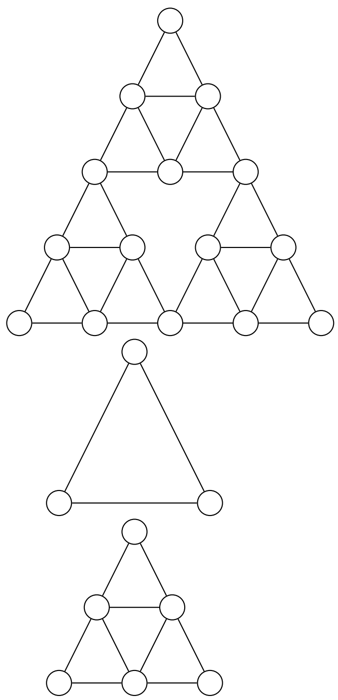
Summing all contributions to the spectrum at level m: $2(3^{m-1} - 1)$ doubled eigenfunctions through spectral decimation. 1 constant 0-series eigenfunction. $\frac{3^{m-1}+3}{2}$ continued 6-series eigenfunctions from level $m-1$. $\frac{3^m+3}{2}$ born 6-series eigenfunctions at level m. $\frac{3^{m-1}-1}{2}$ born 5-series eigenfunctions at level m.

Adding these gives the total:

$$2(3^{m-1}-1)+1+\frac{3^{m-1}+3}{2}+\frac{3^m+3}{2}+\frac{3^{m-1}-1}{2}=\frac{3^{m+1}+3}{2}$$

This confirms the expected dimension of the Neumann spectrum on Γ_m.

Generic Γ_m Diagrams $(m = 0, 1, 2)$



The spectrum for Δ_1 and Δ_2

The Neumann spectrum for Δ_1 is $\{0, 3, 6\}$ with multiplicities of $(1, 2, 3)$ respectively.

The Neumann spectrum for Δ_2 is $\{0, \frac{5-\sqrt{13}}{2}, 3, \frac{5+\sqrt{13}}{2}, 5, 6\}$ with multiplicities $(1, 2, 3, 2, 1, 6)$ respectively.