Resistance Metric on Random Point Approximations of the Sierpinski Gasket and Nonlinear Dimension Reduction

Laplacian Eigenmaps are an important technique for nonlinear dimensionality reduction in machine learning; it is assumed that a data set lies on a low dimensional set in a high dimensional vector space, and eigenfunctions of a graph Laplacian on these points are used as coordinates to map into a low dimensional vector space in a way that retains geometric structure [3]. To prove that such maps preserve geometric information, most of existing results assume that the underlying structure of the data set is a manifold with some smoothness. We are instead interested in the case of non-smooth structures such as fractals, and consider the Sierpinski Gasket as a foundational example. The Sierpinski Gasket is a highly studied fractal in the field of fractal analysis, an expanding field studying analysis and probability on non-smooth structures.

Building on previous REU work [1,2] which studied random graph constructions on a fractal unit interval model, we study graphs arising from independent Poisson random point distributions [4] on cells of approximations to the Sierpinski Gasket. These produce random resistances on graphs associated to the Hanoi group. In order to study the expected behavior of eigenmaps in this setting we investigate the expected resistance and the variance of the resistance, as well as the convergence of the resistance metrics on these Hanoi graphs as we increase the level of approximation to the Sierpinski Gasket. We give formulas for the expected value and variance of resistance across boundary points, with bounded error terms. In addition to our theoretical results we use numerical simulations to illustrate the convergence behavior of Laplacian eigenmaps for this model, with a future goal of computing green's function via resistance forms, and applying results to more general spaces.

[1] Akwei, Bernard, Bobita Atkins, Rachel Bailey, Ashka Dalal, Natalie Dinin, Jonathan Kerby-White, Tess McGuinness et al. "Convergence, optimization and stability of singular eigenmaps." arXiv:2406.19510 (2024).
[2] Akwei, Bernard, Elijah Anderson, Faye Castro, Hank Ewing, Luke G. Rogers, Alexander Teplyaev. "Singular Laplacian eigenmaps via one-dimensional Green's function."(in preparation, 2025)
[3] Belkin, Mikhail, and Partha Niyogi. "Laplacian eigenmaps for dimensionality reduction and data representation." Neural computation 15, no. 6 (2003): 1373-1396.
[4] Armstrong, Scott, and Raghavendra Venkatraman. "Quantitative homogenization and large-scale regularity of Poisson point clouds." arXiv:2309.12900 (2023).

REU Team members:
Ranjeev Iyer, UConn
Ben Roark, Coe College
Ty Wick, Saint Michael's College

Research collaborators at UConn:
Bernard Akwei
Luke Rogers
Alexander Teplyaev