

Spectral Properties of the Conformally Invariant Laplacian on the Basilica Group

The Basilica group is a self-similar group generated by a 3-state automaton first introduced by Grigorchuk and Zuk in [1]. Their results and later work of Bartholdi and Virag [2] established it as the first example of an amenable but not subexponentially amenable group. In the work of Nekrashevych [3] it occurs as the iterated monodromy group of the quadratic polynomial $z^2 - 1$, the simplest nontrivial example in his theory of iterated monodromy groups.

We consider the spectral properties of the Schreier graphs of the Basilica group. These were first considered in [4], where it was shown that there is a family of weighted Laplacians on these graphs whose spectra are invariant under a two-dimensional dynamical system. Analysis of the spectra requires more investigation, but many properties of the conformally invariant Laplacian on a family of subgraphs were established in [5] and some similar results have been announced by Dang, Grigorchuk and Lyubich using their recently-introduced notion of spectral currents.

Our work involves generalizing the results of [5] to the conformally invariant Laplacians on the Schreier graphs of the Basilica group. These correspond to a natural subfamily of the Laplacians from [4]. We introduce a recursion for the characteristic polynomial of the conformally invariant Laplacian, use geometric features to describe the multiplicities of eigenvalues that appear in earlier levels of the spectrum, and construct localized and non-localized eigenfunctions. Our geometric analysis shows that the spectra of these operators are invariant under a simpler dynamical system from that considered in [4], and hence give a different approach to the results of Dang, Grigorchuk and Lyubich [6].

References:

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Structure and Dynamics of Laplacian Eigenfunctions on the Basilica Fractal

We study the Basilica group, a self-similar group generated by a 3-state automaton. Grigorchuk and Zuk [1], and later Bartholdi and Virag [2], established the group as the first example of an amenable but not subexponentially amenable group, and in the work of Nekrashevych [3], it occurs as the simplest nontrivial example in his theory of iterated monodromy groups.

In this talk, we investigate the spectral properties of the Schreier graphs of the Basilica group. These were first considered in [4], which revealed a family of weighted Laplacians on these graphs whose spectra are invariant under a two-dimensional dynamical system. A recent analysis of this dynamical system by Dang, Grigorchuk, and Lyubich [5] yields results on limiting properties of these spectra, obtained with respect to the conformally-invariant Laplacian in [6]. Our interest is in describing the structure of the eigenfunctions for this sequence of graphs, for which purpose we note that the dynamics from [4] correspond to a version of the notion of spectral self-similarity used by [7]. We develop an alternative description of the dynamical system in [4] and study the corresponding dynamics for the eigenfunctions. Moving forward, we aim to determine whether certain orbital Schreier graphs considered in [8] have singular or absolutely continuous spectra.

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