Structure and Dynamics of Laplacian Eigenfunctions on the Basilica Fractal

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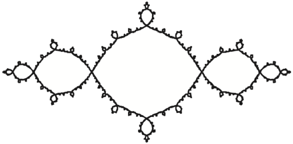


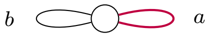
Background

History and Motivation

We expand on work by Grigorchuk and Żuk, who studied the Schreier graphs of the Basilica group [4, 5].

- Amenability and subexponential growth of the Basilica group
- Weighted Laplacians on these graphs with spectra invariant under a two-dimensional dynamical system
- We develop an alternative description of the dynamical system in [5] and study the corresponding dynamics for the eigenfunctions.

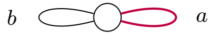
Background: Graph Approximations of the Basilica Set

Γ_0: 

a-edges (pink) have weight ω_0; b-edges (black) have weight 1

Background: Graph Approximations of the Basilica Set

Γ_0:



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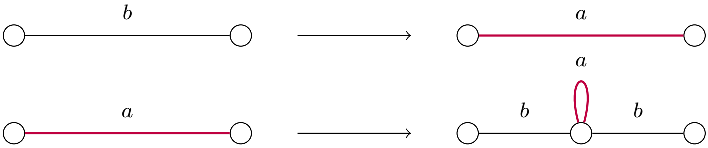
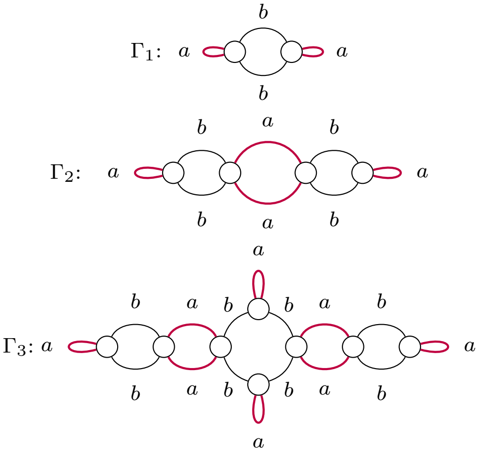
Replacement Rules



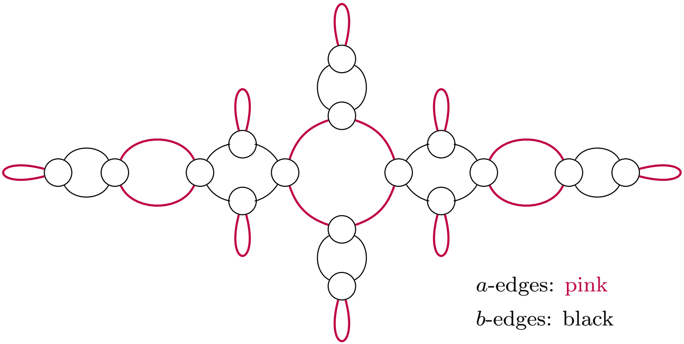
Background: Graph Approximations of the Basilica Set



a-edges (pink) have weight $\omega_n \in \mathbb{R}^+$; b-edges (black) have weight 1

Background: Graph Approximations of the Basilica Set

Γ_4:



a-edges: pink

b-edges: black

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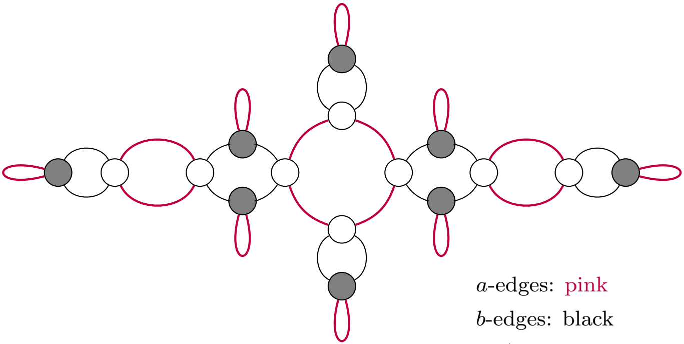

a-edges: pink

b-edges: black

$V_n \setminus V_{n-1}$ vertices: gray

The Graph Laplacian

The graph Laplacian encodes all information on Γ_n as a $2^n \times 2^n$ matrix:

$$\Delta_n = D_n - A_n,$$

where $D_n = (2 + 2\omega_n)I$ is the diagonal matrix and $A_n = (a_{ij})$ is the adjacency matrix, with entries

$$a_{ij} = \begin{cases} k\,\omega_n \text{ if } i \sim j \text{ (via } k\, a\text{-edges);} \\ k \text{ if } i \sim j \text{ (via } k\, b\text{-edges);} \\ 0 \text{ if } i \not\sim j. \end{cases}$$

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Definition (Laplacian Eigenfunction)

For all $v \in V_n$, a Laplacian eigenfunction u_n on Γ_n satisfies

$$\Delta_n u_n(v) = \lambda_n u_n(v).$$

The Graph Laplacian Acting on Functions

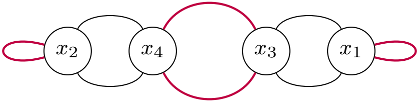
For a single vertex $v \in V_n$ and a function u on Γ_n,

$$\Delta_n u(v) = \omega_n \sum_{x \underset{a}{\sim} v} \big(u(v) - u(x)\big) + \sum_{x \underset{b}{\sim} v} \big(u(v) - u(x)\big).$$

Example: On Γ_2 the graph Laplacian on a function u is

$$\Delta_2 u = \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2+2\omega_2 & -2\omega_2 \\ 0 & -2 & -2\omega_2 & 2+2\omega_2 \end{bmatrix} \begin{bmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \end{bmatrix}.$$

Γ_2:



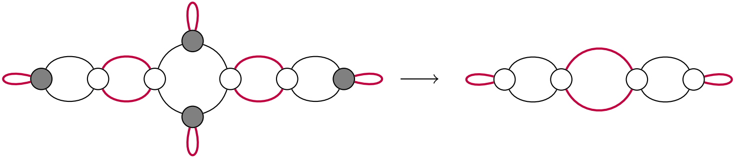
Results

Constructing Eigenfunctions

Theorem (Restricting u_n to obtain u_{n-1})

Suppose Γ_n has weight $\omega_n \neq 0$ and $\lambda_n \neq 2$, where u_n satisfies $\Delta_n u_n = \lambda_n u_n$. Then, restricting u_n to the vertices V_{n-1} produces an eigenfunction on Γ_{n-1}, with

$$\lambda_{n-1} = \frac{4\lambda_n - \lambda_n^2}{\omega_n\,(2-\lambda_n)} \quad \textit{and} \quad \omega_{n-1} = \frac{1}{\omega_n\,(2-\lambda_n)}.$$



Removing $V_3 \setminus V_2$ (gray) to obtain Γ_2

Constructing Eigenfunctions (Proof)

We construct Δ_n recursively [5]. Let $a_0 = b_0 = 1$,

$$a_n = \begin{bmatrix} I_{n-1} & 0 \\ 0 & b_{n-1} \end{bmatrix}, b_n = \begin{bmatrix} 0 & I_{n-1} \\ a_{n-1} & 0 \end{bmatrix}.$$

Then, $\Delta_n = 2 + 2\omega_n - \omega_n \left(a_n + a_n^{-1}\right) - \left(b_n + b_n^{-1}\right)$ and $\Delta_n - \lambda_n =$

$$\begin{bmatrix} 2 - \lambda_n & -\left(a_{n-1}^{-1} + 1\right) \\ -\left(a_{n-1} + 1\right) & 2 + 2\omega_n - \omega_n \left(b_{n-1} + b_{n-1}^{-1}\right) - \lambda_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

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Write the eigenfunction $u_n = \begin{bmatrix} \widehat{u_n} \\ u_{n-1} \end{bmatrix}$, where $\widehat{u_n} = u_n\Big|_{V_n \setminus V_{n-1}}$ and $u_{n-1} = u_n\Big|_{V_{n-1}}$. Then

$$\left(\Delta_n - \lambda_n\right) u_n = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \widehat{u_n} \\ u_{n-1} \end{bmatrix} = 0.$$

Constructing Eigenfunctions (Proof)

The Schur complement of A, $D - CA^{-1}B$, "deletes" vertices $V_n \setminus V_{n-1}$:

$$\left(D - CA^{-1}B\right) u_{n-1} = 0$$

$$\left(2 + \frac{2}{\omega_n(2-\lambda_n)} - \frac{a_{n-1} + a_{n-1}^{-1}}{\omega_n(2-\lambda_n)} - \left(b_{n-1} + b_{n-1}^{-1}\right) - \frac{4\lambda_n - \lambda_n^2}{\omega_n(2-\lambda_n)}\right) u_{n-1} = 0.$$

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Compare to the eigenvalue equation on Γ_{n-1}:

$$\left(2 + 2\omega_{n-1} - \omega_{n-1}\left(a_{n-1} + a_{n-1}^{-1}\right) - \left(b_{n-1} + b_{n-1}^{-1}\right) - \lambda_{n-1}\right) u_{n-1} = 0.$$

Thus, u_{n-1} is an eigenfunction for Δ_{n-1} with

$$\omega_{n-1} = \frac{1}{\omega_n\left(2-\lambda_n\right)}, \quad \lambda_{n-1} = \frac{4\lambda_n - \lambda_n^2}{\omega_n\left(2-\lambda_n\right)}$$

□

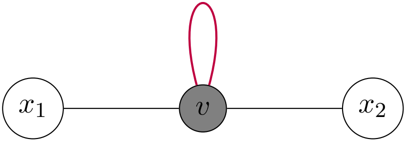
Constructing Eigenfunctions

Theorem (Building u_{n+1} from u_n)

Let u_n satisfy $\Delta_n u_n(v) = \lambda_n u_n(v)$ for all $v \in V_n$, and let $\lambda_{n+1} \neq 2$ and ω_{n+1} be written in terms of λ_n and ω_n using the eigenvalue-weight dynamics. Then u_n extends to V_{n+1} by:

$$u_{n+1}(v) = \frac{1}{2 - \lambda_{n+1}} \sum_{x \sim v} u_n(x),\ v \in V_{n+1} \setminus V_n.$$

This creates an eigenfunction u_{n+1} satisfying $\Delta_{n+1} u_{n+1} = \lambda_{n+1} u_{n+1}$.



Cell in Γ_{n+1}

$$u_{n+1}(v) = \frac{u_n(x_1) + u_n(x_2)}{2 - \lambda_n}$$

Characteristic Polynomial Recursion

Corollary

Let $\Psi_{n-1}(\omega_{n-1}, \lambda_{n-1})$ be the characteristic polynomial of Δ_{n-1} on Γ_{n-1}, with $\omega_{n-1} \neq 0$. Then the characteristic polynomial of Δ_n on Γ_n is given by

$$\Psi_n\left(\omega_n, \lambda_n\right) = \left(\omega_n(2-\lambda_n)\right)^{2^{n-1}} \Psi_{n-1}\left(\frac{1}{\omega_n(2-\lambda_n)}, \frac{4\lambda_n - \lambda_n^2}{\omega_n(2-\lambda_n)}\right).$$

Motivation: Grigorchuk and Żuk prove analogous results for the adjacency matrix with their dynamics, which we adapt to Γ_n [5].

Inverting Dynamics

$$\lambda_{n-1} = \frac{4\lambda_n - \lambda_n^2}{\omega_n\,(2-\lambda_n)},\ \ \omega_{n-1} = \frac{1}{\omega_n(2-\lambda_n)}$$

$$\lambda_n = 2 \pm \sqrt{4 - \frac{\lambda_{n-1}}{\omega_{n-1}}},\ \ \omega_n = \frac{\mp 1}{\omega_{n-1}\sqrt{4-\frac{\lambda_{n-1}}{\omega_{n-1}}}}$$

- With $(\lambda_n, \omega_n) \to (\lambda_{n-1}, \omega_{n-1})$, need $\lambda_n \neq 2$
- With $(\lambda_{n-1}, \omega_{n-1}) \to (\lambda_n, \omega_n)$, need $4 > \frac{\lambda_{n-1}}{\omega_{n-1}}$

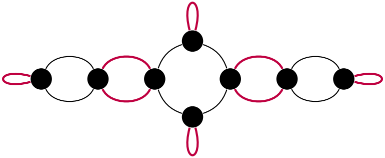
Recursive Eigenfunction Construction (Reminder)

$$u_{n+1}(v) = \frac{1}{2-\lambda_{n+1}} \sum_{x \sim v} u_n(x),\ v \in V_{n+1} \setminus V_n$$

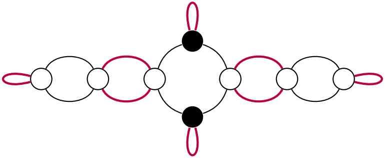
The 0-Series and 2-Series

Lemma (0-Series and 2-Series)

For any λ_n there exists some $m \leq n$ such that λ_n recurses down to $\lambda_m = 0$ or $\lambda_m = 2$.



0-Series Eigenfunction ($n = 3$)
(non-zero points in black)



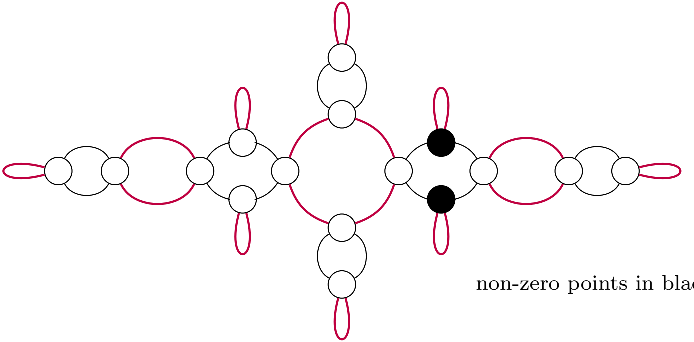
2-Series Eigenfunction ($n = 3$)
(non-zero points in black)

- **0-series**: At some $0 \leq m < n$, $\lambda_m = 0$; V_m is the constant function
- **2-series**: At some $3 \leq m < n$, $\lambda_m = 2$; V_{m-1} is the zero function

2-series Eigenfunctions

Lemma ($\lambda_n = 2$ Multiplicity)

The multiplicity of 2 as an eigenvalue of Γ_n is $\dfrac{2^{n-1} + 3 + (-1)^{n-2}}{6}$.



non-zero points in black

Eigenfunction Construction Theorem

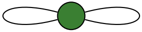
Theorem

Given an eigenvalue λ_n of Δ_n with a specific ω_n, its corresponding eigenfunction can be constructed as above.

1. Use $(\lambda_n, \omega_n) \to (\lambda_{n-1}, \omega_{n-1})$ recursions to get back to $\lambda_m = 0$ or $\lambda_m = 2$ $(m \leq n)$ noting $+, -$ patterns.
2. Use $(\lambda_{n-1}, \omega_{n-1}) \to (\lambda_n, \omega_n)$ recursions and recursive construction equation to build eigenfunctions up from Γ_m, following $+, -$ choices.

Behavior of Eigenfunctions

positive values in green, negative values in red



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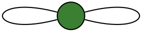
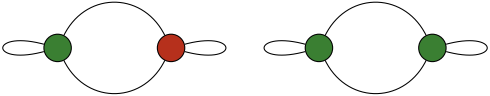


+  −

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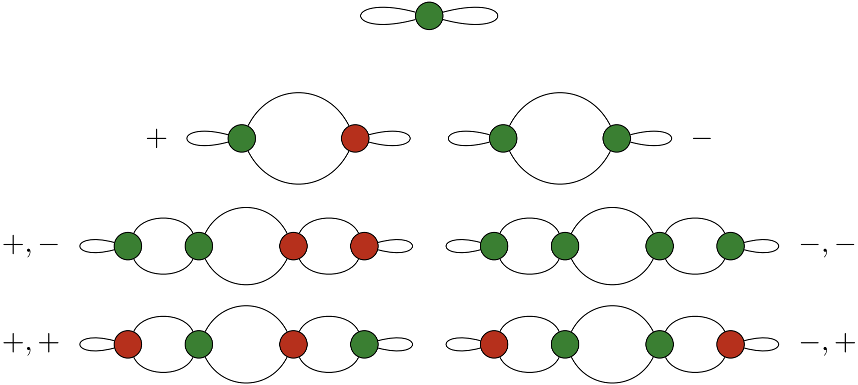

Recursive Eigenfunction Construction Conjecture

$$\lambda_n = 2 \pm \sqrt{4 - \frac{\lambda_{n-1}}{\omega_{n-1}}}, \quad \omega_n = \frac{\mp 1}{\omega_{n-1}\sqrt{4 - \frac{\lambda_{n-1}}{\omega_{n-1}}}}$$

Conjecture (Recursively Building Eigenfunctions)

When recursively building eigenfunctions, the function exists if and only if its corresponding $+/-$ sequence

1. *Never has a $+++$ subsequence, and*
2. *If it has a $++-$ subsequence, the subsequence is followed by an odd number of nonconsecutive $+$'s.*

Example:

Good Sequence:	Bad Sequence:
$++-+-+-+$	$++-+--+-$

Conclusion and Next Steps

Summary

- Expanded on recursive dynamics from [5]
- New recursive relationships for Laplacian eigenfunctions
- Spectral properties of eigenfunctions between levels

Next Steps & Further Research

- Proving conjectures
- Further research on dynamics
  - Fixed points
  - Forbidden regions on $\lambda - \omega$ plane
- Determine whether a variant of Γ_∞ (one-ended blowup) has continuous spectrum [1, 2]

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Thank you!