Spectrum of the Basilica Schreier Graphs

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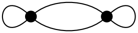
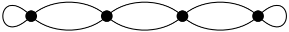
Defining Γ_n Graphs

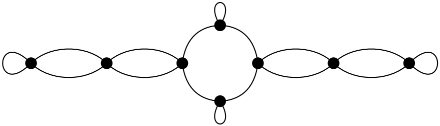
Γ_1:



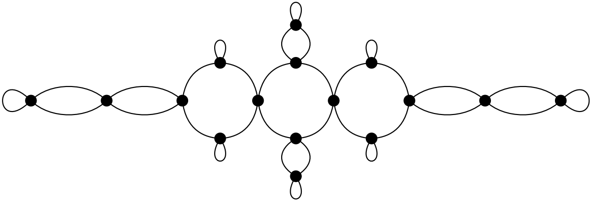
Γ_2:



Γ_3:

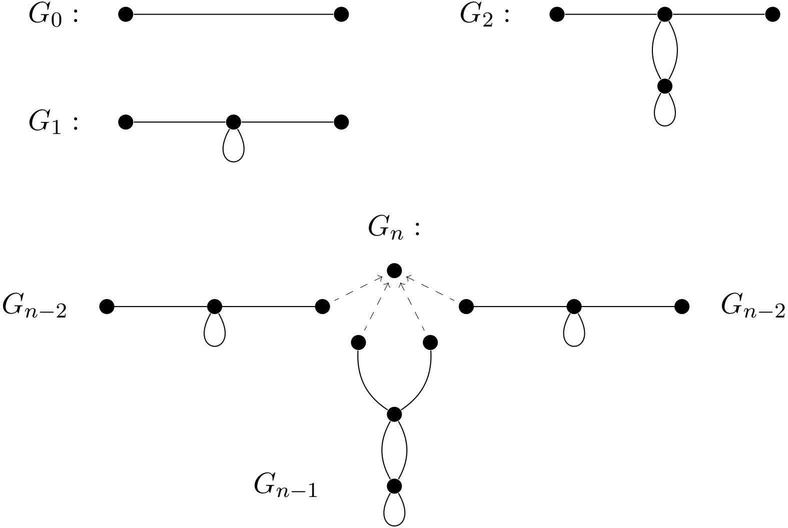


Γ_4:



Defining the Basilica

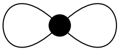
We look at the following construction:



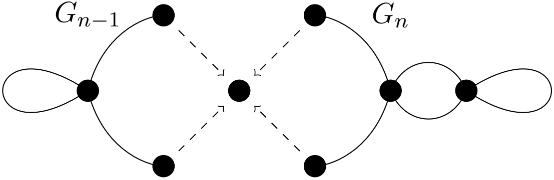
Defining Γ_n Graphs I

We can construct Γ graphs from G graphs as follows:

Γ_0 :

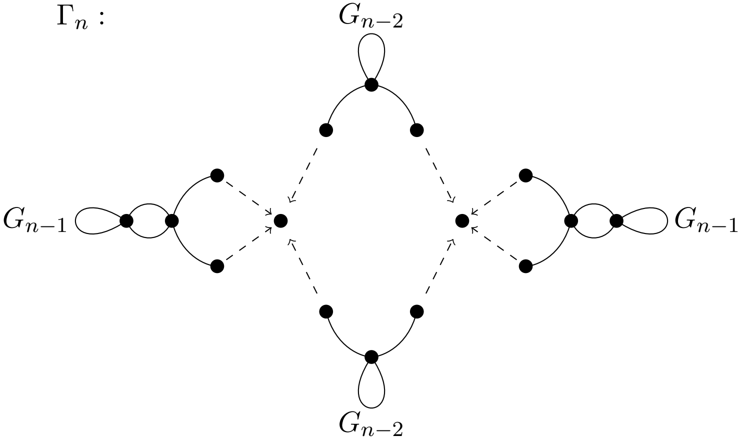


Γ_n :



Defining Γ_n Graphs II

An equivalent construction of Γ_n is:



The Basilica

This construction of Γ graphs are the finite Schreier graphs of the Basilica Group. The Basilica fractal has been studied by

- Nekrashevych in the theory of iterated monodromy groups [4]
- Bartholdi in the theory of amenable groups [1]
- Grigorchuk and Zuk as a 3-state automata [3]
- Nekrashevych as the Julia Set of $z^2 - 1$ [4]

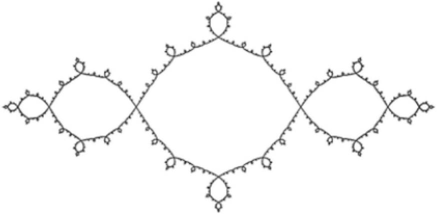


Figure: The Basilica Julia set, the Julia set of $z^2 - 1$.

Background Definitions I

Definition (Graph Laplacian)

The graph Laplacian encodes all information on the graph Γ_n as a matrix

$$\Delta(\Gamma_n) = D_n - A_n.$$

$D_n = 4I$ is the degree matrix and $A_n = (a_{ij})$ the adjacency matrix.

$$a_{i,j} = \begin{cases} k \text{ if } i \sim j \text{ (via } k \text{ edges);} \\ 0 \text{ if } i \not\sim j \end{cases}$$

Background Definitions II

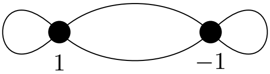
Definition (Eigenfunctions and Eigenvalues)

λ is an eigenvalue of the graph Laplacian with eigenfunction f if

$$\Delta\left(\Gamma_n\right) f = \lambda f$$

For some $\lambda \in \mathbb{R}_{\geq 0}$ and $f \not\equiv 0$.

$\lambda = 4$



At a specific vertex, the sum of the edge differences must be equal to $\lambda \cdot f(x)$.

$$\lambda \cdot f(x) = \sum_{x \sim y} (f(x) - f(y))$$

Background Definitions III

Definition (Dirichlet Eigenfunctions and Eigenvalues of G_n)

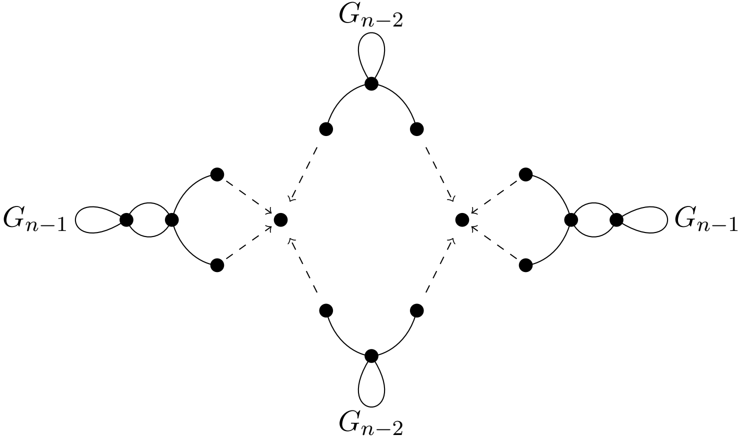
λ is a Dirichlet eigenvalue of $\Delta(G_n)$ with associated Dirichlet eigenfunction f if $f|_{\partial G_n} = 0$ and λ and f are an eigenvalue/eigenfunction pair of $\Delta(G_n \setminus \partial G_n)$.

Definition (spec and Dirspec)

$spec(\Delta(\Gamma_n))$ is the set of eigenvalues of $\Delta(\Gamma_n)$ including multiplicity
$Dirspec(\Delta(G_n))$ is the set of Dirichlet eigenvalues of $\Delta(G_n)$ including multiplicity

G_n to Γ_n Connections

- Previous work in [2] has fully understood the spectrum of the Dirichlet Laplacian on the G_n graphs.
- We use this previous work to understand the spectrum of the Laplacian on the Γ_n graphs.



Motivations Summary

- The Γ_n graphs are the finite Schreier graphs of the Basilica group
- We can build the Γ_n graphs from G_n graphs
- We can define the G_n graphs recursively from previous G_n graphs
- The spectrum of the Dirichlet Laplacian on the G_n graphs is understood in previous work
- We extend this previous work to understand the spectrum of the Laplacian on Γ_n

Gluing Maintains Eigenfunctions I

Theorem

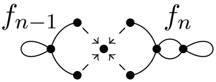
For any $n \in \mathbb{N}$, *if* λ *is an eigenvalue of* $\Delta(\Gamma_n)$ *then* λ *is an eigenvalue of* $\Delta(\Gamma_{n+1})$.

Gluing Maintains Eigenfunctions II

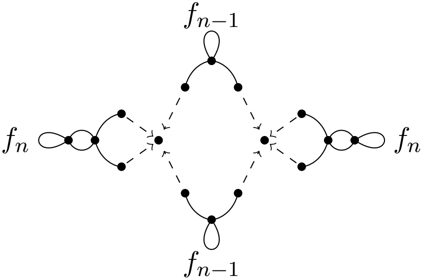
Let λ be an eigenvalue of $\Delta(\Gamma_n)$ with eigenfunction f.

$$f_n = f|_{G_n} \text{ and } f_{n-1} = f|_{G_{n-1}}$$

Γ_n :



Γ_{n+1} :



Thus, using this restriction, it must be that there is an eigenfunction on $\Delta(\Gamma_{n+1})$ with eigenvalue λ.

Definition

λ is born on level n for arbitrary graph T_n if $\lambda \in spec(\Delta(T_n))$, but $\lambda \notin spec(\Delta(T_m))$ for any $m < n$.

We seek a recursion on the characteristic polynomial of $\Delta(\Gamma_n)$.

Definition

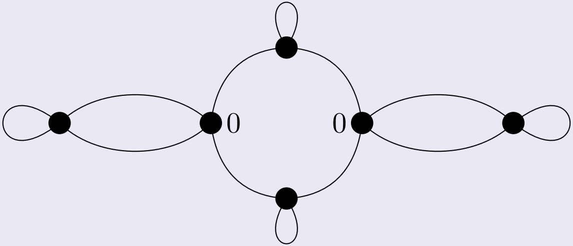
The characteristic polynomial of the graph Laplacian on Γ_n is defined as:

$$P_n = \prod_{\lambda_j \in spec(\Delta(\Gamma_n))} (\lambda - \lambda_j)$$

Definitions of 2-series and 0-series I

Definition (2-series Eigenvalue)

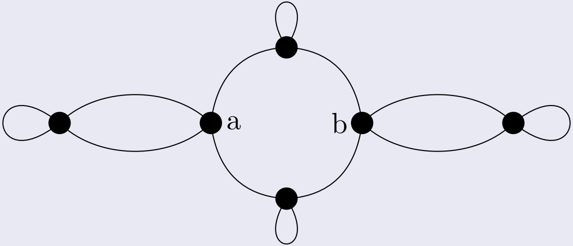
Any eigenvalue of $\Delta(\Gamma_n)$ associated with some eigenfunction having both values to the left and right of the center loop equal to 0.



Definitions of 2-series and 0-series II

Definition (0-series Eigenvalue)

Any eigenvalue of $\Delta\left(\Gamma_n\right)$ associated with some eigenfunction having $a \neq 0$ or $b \neq 0$



Motivation for Characteristic Polynomial Factorization

The characteristic polynomial of $\Delta(G_n \setminus \partial G_n)$ has some factorization, so we expect the characteristic polynomial for Γ_n graphs to look something like

$$P_n = \prod_{k=0}^{n} \psi_k^{\alpha_{n,k}} \upsilon_k^{\beta_{n,k}}$$

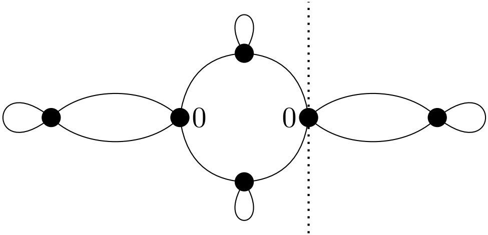
- $\psi_{n,k}$ is the polynomial whose roots are the 0-series eigenvalues born at level k
- $\upsilon_{n,k}$ is the polynomial whose roots are the 2-series eigenvalues born at level k

2-series eigenvalues are in the Dirichlet spectrum of G_n

Lemma

If $\lambda \in spec(\Delta(\Gamma_n))$ and is in the 2-series then λ is in the Dirichlet spectrum of the Laplacian on $G_m \setminus \partial G_m$ for some m.

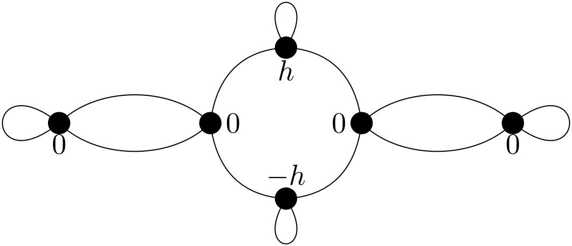
Proof: Let $\lambda \in spec(\Delta(\Gamma_n))$ be an element of the 2-series. At least one of the restrictions to G_n or G_{n-1} is non-zero. Therefore, λ must be a Dirichlet eigenvalue of $\Delta(G_n)$ or $\Delta(G_{n-1})$.



$G_{n-2} \setminus \partial G_{n-2}$ Eigenvalues to Γ_n

Lemma

If λ first appears as an eigenvalue at level $n-2$ on $G_{n-2} \setminus \partial G_{n-2}$, then it first appears as an eigenvalue at level n on Γ_n



2-series eigenvalues are simple on the level they are born

Lemma

If $\lambda \in spec(\Delta(\Gamma_n))$ and is in the 2-series, then λ is simple at the level it was born

This result comes from the fact that λ must be in $\mathrm{Dirspec}(\Delta(G_n))$. λ is simple on G_n at the level it was born, and the Γ_n graphs are built from G_n graphs.

2-series Eigenvalues Multiplicity

Lemma

If $\lambda \in spec(\Delta(\Gamma_n))$ is in the 2-series and λ is born at level k, the multiplicity of λ at level n is $\dfrac{2^{n-k+2} + 3 + (-1)^{n-k+1}}{6}$

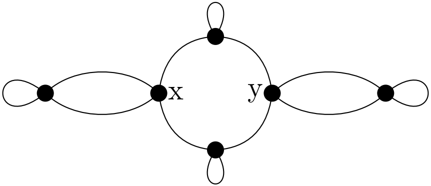
This shown by gluing copies of G_{k-2} graphs together to build Γ_n graphs for any n, and counting the eigenfunctions.

0-series eigenvalues are simple

Lemma

If $\lambda \in spec(\Delta(\Gamma_n))$ and is in the 0-series, then λ is simple at every level n

Proof: Suppose $\lambda \in spec(\Delta(\Gamma_n))$ lies in the 0-series. Then $\lambda \notin \text{Dirspec}(\Delta(G_m))$ for any m. If two eigenfunctions correspond to λ, any linear combination vanishing at a gluing point must be identically zero.



Factorization into γ and ψ

Theorem

The characteristic polynomial of the spectrum of Γ_n is:

$$P_n = \psi_n \gamma_{n-2} \prod_{k=0}^{n-1} \psi_k \gamma_{k-2}^{\sigma_{n-k}}$$

$\sigma_{n-k} = \frac{2^{2+n-k}+3+(-1)^{n-k+1}}{6}$

ψ_k *is the polynomial whose roots are the* 0*-series eigenvalues born at level* k

γ_{k-2} *is the polynomial whose roots are the* 2*-series eigenvalues born at level* k

Motivation for Dynamics

- In previous work [2], there are simple dynamics for the new roots of the characteristic polynomial of $\Delta(G_n \setminus \partial G_n)$
- We find dynamics for the new roots of the characteristic polynomial of $\Delta(\Gamma_n)$

Theorem (Brzoska et al.)

For

$$\zeta_n = \frac{\gamma_n}{\eta_n}, \quad \eta_n = \gamma_{n-1} \sum_{0 \le 2j \le n-4} \gamma_{n-2j-3}^{2^k},$$

then if $n \geq 4$,

$$\zeta_n - 2 = \left(1 + \frac{2}{\zeta_{n-1}}\right)\left(\zeta_{n-2}^2 - 4\right)$$

Z_n Dynamics

Theorem

For

$$\mathrm{Z}_n = \frac{\psi_n \gamma_{n-2}}{\gamma_n}$$

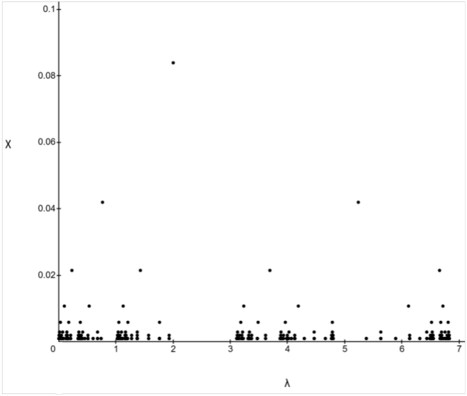
then if $n \geq 5$,

$$\mathrm{Z}_{n-1}\mathrm{Z}_{n-2}\zeta_n\zeta_{n-1}\left(\mathrm{Z}_n - 1\right)\left(\zeta_{n-3}^2 - 4\right) = \left(\mathrm{Z}_{n-2} - 1\right)\left(\zeta_{n-1}^2 - 4\right)\zeta_{n-2}$$

Spectral Measure Graph

$$\chi_n = \sum_{k=0}^{n} \left(\sum_{\{\lambda : \gamma_{k-2}(\lambda)=0\}} \frac{\sigma_{n-k}}{2^n} \delta_\lambda + \sum_{\{\lambda : \psi_n(\lambda)=0\}} \frac{1}{2^n} \delta_\lambda \right)$$

Spectral Measure graph at $n = 10$



KNS Spectral Measure

Theorem

The Kesten-Von-Neumann-Serre (KNS) Spectral Measure of $\Delta\left(\Gamma_n\right)$ *is*

$$\chi = \underset{n\to\infty}{\text{w-lim}}\left(\chi_n\right) = \sum_{k=0}^{\infty} \sum_{\{\lambda : \gamma_{k-2}(\lambda)=0\}} \frac{2}{3} \cdot 2^{-k} \cdot \delta_\lambda$$

Summary

- We understand the spectrum of the Laplacian on the Γ_n graphs
- We used our understanding of the Dirichlet spectrum of G_n and gluing to understand the spectrum of the Laplacian on Γ_n graphs
- We factored the characteristic polynomial of $\Delta(\Gamma_n)$ into a product of 0-series and 2-series terms
- We found dynamics which describe the new factors of the characteristic polynomial of $\Delta(\Gamma_n)$
- We found the KNS spectral measure

Citations

[1] Laurent Bartholdi and Bálint Virág. "Amenability via random walks". In: *Duke Mathematical Journal* 130.1 (Oct. 2005). ISSN: 0012-7094. DOI: 10.1215/s0012-7094-05-13012-5. URL: http://dx.doi.org/10.1215/S0012-7094-05-13012-5.

[2] Antoni Brzoska et al. *Spectral properties of graphs associated to the Basilica group*. 2025. arXiv: 1908.10505 [math.GR].

[3] Rostislav I. Grigorchuk and Andrzej Żuk. "Spectral properties of a torsion-free weakly branch group defined by a three state automaton". In: 2002.

[4] Volodymyr Nekrashevych. "Self-Similar Groups". In: *Mathematical Surveys and Monographs* (2005).