# Random Approximation of the Sierpinski Gasket

Ranju Iyer, Benjamin Roark, Ty Wick
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# Uniform Case Lemma

In the uniform case where each cell has equal points distributed.

**Lemma**

*Starting with $H_n$ reducing to $H_{n-1}$ the resistances $r_{xy}$ are adjusted by a factor of $\frac{5}{3}$*

# Deterministic Case Thm.

**Theorem**

*If starting with graph $H_n$ having edge resistance $r_{xy}$ and reducing to $H_1$ we get the resistance to be $\left(\frac{5}{3}\right)^{n-1} r_{xy}$. From here we can reduce to 2 points, where the effective resistance is $\left(\frac{5}{3}\right)^{n} r_{xy}$.*

# Deterministic Case Cor.

**Corollary**

*Between tail points $p$ and $q$ we find that to get $R_n(p,q) = 1$, we should have $r_{xy} = (\frac{3}{5})^n$, except for the tail weights, which should be $\frac{1}{2}(\frac{3}{5})^n$.*

## Lemma

$$\mathbb{E}r_{xy} = \frac{1}{\lambda^2} + \epsilon, \textit{ where } |\epsilon| \leq \frac{2e^{-\lambda}}{\lambda^2}.$$

$$r_{xy} = \frac{1}{1+X}\frac{1}{1+Y}$$

where $X$ and $Y$ are i.i.d. Poisson random variables.

$$\mathbb{E}r_{xy} = \mathbb{E}\frac{1}{1+X}\frac{1}{1+Y} = (\mathbb{E}\frac{1}{1+X})^2$$

because $X$ and $Y$ are i.i.d. With direct computation, we get

$$\mathbb{E}(\frac{1}{1+X}) = \sum_{n=0}^{\infty} \frac{1}{1+n} \cdot \frac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty} \frac{\lambda^n}{(n+1)!}$$

$$= e^{-\lambda} \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{j!} = \frac{e^{-\lambda}}{\lambda} (\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} - 1) = \frac{1}{\lambda}(1 - e^{-\lambda}).$$

Thus,

$$\mathbb{E}r_{xy} = [\frac{1}{\lambda}(1 - e^{-\lambda})]^2 = \frac{1}{\lambda^2} - \frac{2e^{-\lambda}}{\lambda^2} + \frac{e^{-2\lambda}}{\lambda^2}.$$

We then note that, for $\lambda > 0$,

$$\frac{-2e^{-\lambda}}{\lambda^2} + \frac{e^{-2\lambda}}{\lambda^2} = \frac{e^{-\lambda}}{\lambda^2}(-2 + e^{-\lambda}) \leq \frac{-2e^{-\lambda}}{\lambda^2} \leq \frac{2e^{-\lambda}}{\lambda^2},$$

completing the proof.

## Lemma

$$\mathbb{V}r_{xy} = \frac{1}{\lambda^5} + o$$

Proof: For large $\lambda$,

$$\frac{1}{1+X}\frac{1}{1+Y} \approx \frac{1}{XY}.$$

If we write $X = \lambda + \sqrt{\lambda}Z_X + o$ and $Y = \lambda + \sqrt{\lambda}Z_Y + o$, then

$$\frac{1}{XY} = \frac{1}{\lambda^2(1+\frac{Z_x}{\sqrt{\lambda}}+o)(1+\frac{Z_Y}{\sqrt{\lambda}}+o)} = \frac{1}{\lambda^2(1+\sqrt{\frac{2}{\lambda}}Z+o)}.$$

So,

$$\mathbb{V}(\frac{1}{XY}) = \frac{1}{\lambda^4}\mathbb{V}(1-\sqrt{\frac{2}{\lambda}}Z+o) = \frac{2}{\lambda^5}+o$$

# Laplacian Eigenmaps of $H_n$

**Goals and Outline**

- We aim to provide numerical evidence that the Laplacian eigenmaps of $H_n$ when weighted randomly converge to the eigenmaps when weighted deterministically.

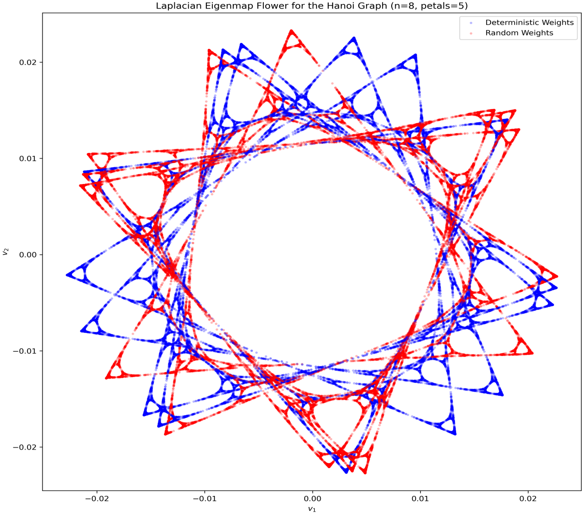
- In the deterministic case we fix all edge resistances $r_{xy} = (3/5)^n$, whereas in the random case we let $r_{xy} = \big[(1+X)(1+Y)\big]^{-1}$ where $X, Y \overset{\text{i.i.d.}}{\sim} \text{Pois}(\lambda)$ and $\lambda(n)$ fulfills $\mathbb{E}(r_{xy}) = (3/5)^n$.

- Given our random and deterministic eigenmaps $A$ and $B$, we expect to see that

$$\|A - B\|_F \to 0, \quad \text{as} \quad n \to \infty$$

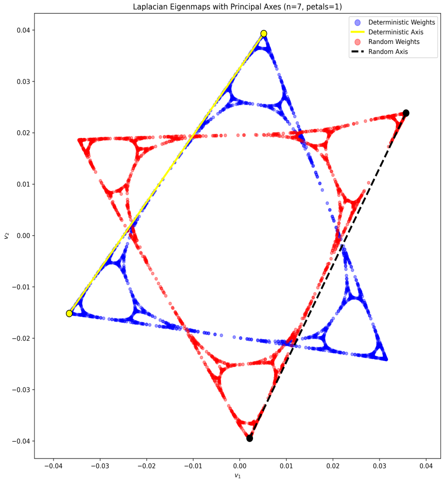
# Initial Computations: The Hanoi Flower

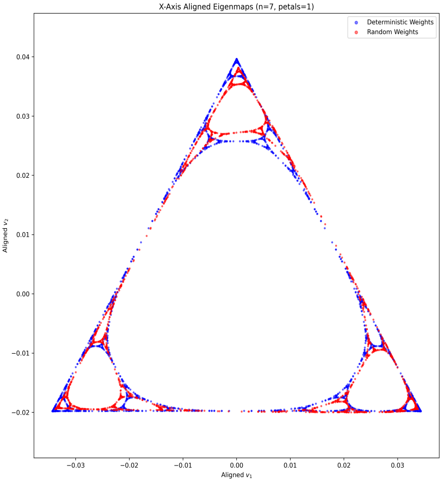
A picture of the initial computation of the Eigenmaps of both the random and deterministic case is shown below.



# Fixing Spurious Rotations

- We use the Convex Hull algorithm to attain the axis of rotation by which we can align the eigenmaps.



# Remaining Work For Numerics

- Although we can align the point clouds visually, we still need to ensure that the indices of the maps coincide with the same points.
- Once this has been completed, we can easily find $\|A - B\|_n$ and report on the convergence of these maps

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